ELSEVIER

Theoretical Computer Science 283 (2002) 243–268

Theoretical Computer Science

www.elsevier.com/locate/tcs

Topological properties of Hausdorff discretization, and comparison to other discretization schemes

Mohamed Tajine *, Christian Ronse

Laboratoire des Sciences de l'Image, de l'Informatique et de la Télédétection (LSIIT, UPRES-A CNRS 7005), Pole API, bd Sebastien Brant, 67400 Illkirch, Cedex, France

Abstract

We study a new framework for the discretization of closed sets and operators based on Hausdorff metric: a Hausdorff discretization of an n-dimensional Euclidean figure F of $\mathbb{R}^n$, in the discrete space $\mathscr{D}_\rho = \rho\mathbb{Z}^n$, is a subset S of $\mathscr{D}_\rho$ whose Hausdorff distance to F is minimal (*ρ can be considered as the resolution of the discrete space $\mathscr{D}_\rho$*); in particular such a discretization depends on the choice of a metric on $\mathbb{R}^n$. This paper is a continuation of our works (Ronse and Tajine, J. Math. Imaging Vision 12 (3) (2000) 219; Hausdorff discretization for cellular distances, and its relation to cover and supercover discretization (to be revised for JVCIR), 2000, Wagner et al., An Approach to Discretization Based on the Hausdorff Metric. I. ISMM'98, Kluwer Academic Publishers, Dordrecht, 1998, pp. 67–74), in which we have studied some properties of Hausdorff discretizations of compact sets.

In this paper, we study the properties of Hausdorff discretization for metrics induced by a norm and we refine this study for the class of homogeneous metrics. We prove that for such metrics the popular covering discretizations are Hausdorff discretizations. We also compare the Hausdorff discretization with the Bresenham discretization (Bresenham, IBM Systems J. 4 (1) (1965) 25). Actually, we prove that the Bresenham discretization of a straight line of $\mathbb{R}^2$ is not always a good discretization relatively to the Hausdorff metric. This result is an extension of Tajine et al. (Hausdorff Discretization and its Comparison with other Discretization Schemes, DGCI'99, Paris, Lecture Notes in Computer Sciences Vol. 1568, Springer, Berlin, 1999, pp. 399–410), in which we prove the same result for a segment of $\mathbb{R}^2$. Finally, we study how some topological properties of the Euclidean plane $\mathbb{R}^2$ are translated in discrete space for Hausdorff discretizations. Actually, we prove that a Hausdorff discretization of a connected closed set is 8-connected and its maximal Hausdorff discretization is 4-connected for homogeneous metrics.

Keywords: Bresenham discretization; Connected; n-connected; Covering discretization; Discrete space; Discretization of operators; Hausdorff metric; Hausdorff discretization; Homogeneous metric

* Corresponding author.
E-mail addresses: tajine@dpt-info.u-strasbg.fr (M. Tajine), ronse@dpt-info.u-strasbg.fr (C. Ronse).

0304-3975/02/$ - see front matter © 2002 Elsevier Science B.V. All rights reserved.
PII: S0304-3975(01)00082-2

1. Introduction

Let F be a non-empty closed subset of the metric space $(\mathbb{R}^n, d)$, a subset S of the discrete space $\mathscr{D}_\rho = \rho\mathbb{Z}^n$ is a Hausdorff discretization in $\mathscr{D}_\rho$ of F if S minimizes the Hausdorff distance to F. We characterize the set $\mathscr{M}_H(F, \rho)$ of subsets of $\mathscr{D}_\rho$ which are Hausdorff discretizations of a non-empty closed set F. We prove that a Hausdorff discretization of a non-empty closed set F converges to F when the resolution ρ of the discrete space converges to 0, as in [6]. Based on this new framework of discretization, we propose a new discretization of operators on the set of non-empty closed sets, and we prove that the discretization of an operator Φ converges to Φ for the set of non-empty compact sets, when the resolution of the discrete space converges to 0, and we prove that some Hausdorff discretizations of a dilation by a compact set are a dilation by finite sets. We refine the study of Hausdorff discretization for the class of homogeneous metrics. We investigate the relationship between Hausdorff discretizations and covering discretizations. We also compare the Hausdorff and Bresenham discretizations [3]. Actually, we prove that the Bresenham discretization of a straight line of $\mathbb{R}^2$ is not always a good discretization relatively to a Hausdorff metric. This result is an extension of the result [25] in which we proved the same thing for a segment of $\mathbb{R}^2$.

Finally, we study the transfer of some topological properties of Hausdorff discretizations between the Euclidean plane $\mathbb{R}^2$ and discrete space for homogeneous metrics. Actually, we prove that Hausdorff discretization of a connected closed set is 8-connected, and that its maximal Hausdorff discretization is 4-connected. The control of transfer of the topological properties from Euclidean spaces to discrete spaces is very important in image processing and image synthesis for building robust algorithms in these fields.

The topological properties of the supercover discretization have been studied by several people: Schmitt [21] shows that, given a compact set K of $\mathbb{R}^2$, under some conditions on K, points can be removed from its supercover discretization $\Delta_{\mathscr{S}\mathscr{C}}(K)$, in such a way that the remaining subset of points is 'homotopically' equivalent to K. Latecki [16] gives sufficient conditions on compact set K, under which $\Delta_{\mathscr{S}\mathscr{C}}(K)$ is 'topologically' equivalent to K.

In all the following, we consider $(\mathbb{R}^n, d)$ as a Euclidean space where d is a metric induced by a norm, and we consider $\mathscr{D}_\rho = \rho\mathbb{Z}^n$ as a discrete space where $\rho \in \mathbb{R}^+$ is the resolution of this discrete space.

Also, for simplifying the notation in the notions which depend on the metric d, we do not refer explicitly to d, except when there is an ambiguity.

This paper is divided into six sections. In Section 2 we present some classical notions of topological space, metric space and Hausdorff metric, and we introduce some new metric notions. In Section 3, we study the problem of discretization with Hausdorff metrics, and we introduce a new discretization: *Hausdorff discretization*. We also investigate the discretization of operators based on the Hausdorff discretization. In Section 4, we refine the study of the Hausdorff discretizations for homogeneous metrics and we investigate the relationship between Hausdorff discretizations and covering discretizations on the one hand, and between Hausdorff discretizations and Bresenham discretization on the other hand. In Section 5, we study some topological properties of Hausdorff discretizations in the plane. The last section is a conclusion.

2. Topological space, metric space, Hausdorff metric and discrete connectivity

This section contains some classical notions of topological space, metric space, normed space and Hausdorff metric. The proofs of the classical results used in this section can be found (for example) in [2, 8, 14].

2.1. Topological space

Definition 1. Given a set $\mathscr{E}$ and subset $\mathscr{T}$ of $\mathscr{P}(\mathscr{E})$, $\mathscr{T}$ is called a *topology* on $\mathscr{E}$ if $\emptyset$ and $\mathscr{E}$ are members of $\mathscr{T}$, and if every finite intersection as well as every union of members of $\mathscr{T}$ is again a member of $\mathscr{T}$. The member of $\mathscr{T}$ is called the *open set* of the topology and the couple $(\mathscr{E},\mathscr{T})$ is called a *topological space*.

A subset F of $\mathscr{E}$ is called *closed set* if $E \setminus F$ is an open set ($(E \setminus F)\in\mathscr{T}$). $\mathscr{F}(\mathscr{E})$ is the set of closed sets of $\mathscr{E}$ and $\mathscr{F}'(\mathscr{E})$ is the set of non-empty closed sets of $\mathscr{E}$ ($\mathscr{F}'(\mathscr{E})=\mathscr{F}(\mathscr{E})\setminus\{\emptyset\}$).

If $X\subseteq\mathscr{E}$, then the set $\mathscr{T}'=\{O\cap X \mid O\in\mathscr{T}\}$ is called the *topology induced* by $\mathscr{T}$ on X or the relative topology on X (*associated with* $\mathscr{T}$) (($X,\mathscr{T}'$) *is a topological space*).

Definition 2. Let $(\mathscr{E},\mathscr{T})$ be a topological space. $\mathscr{E}$ is called a *connected space* if the only sets which are open and closed are $\emptyset$ and $\mathscr{E}$. A subset X of $\mathscr{E}$ is called a *connected set* if it is a connected space relatively to the topology induced by $\mathscr{T}$ on X. If X is a maximal (relatively to the set inclusion) connected subset of $\mathscr{E}$, then X is called a *connected component* of $\mathscr{E}$.

Property 3. *Let* $(\mathscr{E},\mathscr{T})$ *be a topological space. The following assertions are equivalent*:
- $\mathscr{E}$ *is connected.*
- *If* $\mathscr{E}=\bigcup_{i\in I} O_i$ *where* I *is a finite set, each* O_i *is an open set and* $O_i\cap O_{i'}=\emptyset$ *for all* $i\neq i'$, *then there exists* $j\in I$ *such that* $O_j=\mathscr{E}$ *and* $O_i=\emptyset$ *for all* $i\neq j$.
- *If* $\mathscr{E}=\bigcup_{i\in I} F_i$ *where* I *is a finite set, each* F_i *is a closed set and* $F_i\cap F_{i'}=\emptyset$ *for all* $i\neq i'$, *then there exists* $j\in I$ *such that* $F_j=\mathscr{E}$ *and* $F_i=\emptyset$ *for all* $i\neq j$.

Definition 4. Let $(\mathscr{E},\mathscr{T})$ and $(\mathscr{E}',\mathscr{T})$ be two topological spaces and let f be a function from $\mathscr{E}$ to $\mathscr{E}'$. f is *continuous* if $f^{-1}(O')\in\mathscr{T}$ for every $O'\in\mathscr{T}'$, where $f^{-1}(O')=\{x\in\mathscr{E} \mid f(x)\in O'\}$.

Definition 5. Let $\mathscr{E}$ be a topological space. A *path* in $\mathscr{E}$ from x to y is a continuous map $u:[0,1]\to\mathscr{E}$ such that $u(0)=x$ and $u(1)=y$ where $[0,1]$ is considered with the usual topology.

Definition 6. A topological space $\mathscr{E}$ is called *arcwise-connected* if for all x, y in $\mathscr{E}$, there exists a path in $\mathscr{E}$ from x to y. A subset E of $\mathscr{E}$ is called an *arcwise-connected component* of $\mathscr{E}$ if E is a maximal (relatively to the set inclusion) *arcwise-connected* subset of $\mathscr{E}$.

Property 7. *If a topological space is arcwise-connected then it is connected.*

Definition 8. Let $(\mathscr{E}, \mathscr{T})$ be a topological space. $\mathscr{E}$ is called a *compact space* if whenever $\mathscr{E} = \bigcup_{i \in I} O_i$, where O_i is an open set for every $i \in I$, then there exists a finite subset J of I such that $\mathscr{E} = \bigcup_{j \in J} O_j$. A subset X of $\mathscr{E}$ is called a *compact set* if it is a compact space relatively to the topology induced by $\mathscr{T}$ on X.

Property 9. *Let $(\mathscr{E}, \mathscr{T})$ be a compact topological space. If F is a closed set in $(\mathscr{E}, \mathscr{T})$, then F is a compact set.*

2.2. Metric space and normed vector space

Definition 10. A function d from $\mathscr{E} \times \mathscr{E}$ to $\mathbb{R}^+$ is called a metric on $\mathscr{E}$ if:
- $\forall x, y \in \mathscr{E},\ d(x, y) = 0 \Leftrightarrow x = y$,
- $\forall x, y \in \mathscr{E},\ d(x, y) = d(y, x)$ and
- $\forall x, y, z \in \mathscr{E},\ d(x, z) \leqslant d(x, y) + d(y, z)$.

The couple $(\mathscr{E}, d)$ is called a metric space.

Definition 11. Let $(\mathscr{E}, d)$ be a metric space and let $p \in \mathscr{E}$ and $r \in \mathbb{R}^+$,

$$\mathscr{B}_r^d(p) = \{x \in \mathscr{E} \mid d(x, p) \leqslant r\}.$$

$\mathscr{B}_r^d(p)$ is called the *ball of center p and of radius r* relatively to the metric d.

Remark. If d is a metric on $\mathscr{E}$, then $\mathscr{T}(d) = \{O \in \mathscr{P}(\mathscr{E}) \mid \forall x \in O, \exists r > 0, \mathscr{B}_r^d(x) \subset O\}$ is a topology on $\mathscr{E}$. $\mathscr{T}(d)$ is called the topology induced by d on $\mathscr{E}$.

In the following, all topological notions in a metric space $(\mathscr{E}, d)$ are considered relatively to the topology $\mathscr{T}(d)$.

Definition 12. Let $(\mathscr{E}, d)$ be a metric space and let $E \subseteq \mathscr{E}$.
- $int(E) = \{p \in E \mid \exists r > 0,\ \mathscr{B}_r^d(p) \subset E\}$, $int(E)$ is called the *interior* of E.
- $cl(E)$ is the intersection of all closed sets containing E, $cl(E)$ is called the *closure* of E.

Property 13. *Let $(\mathscr{E}, d)$ be a metric space and let $K \subseteq \mathscr{E}$; K is a* compact *set if every infinite sequence in K contains a subsequence having a limit in K.*

Definition 14. A metric d on $\mathbb{R}^n$ is said to be *invariant under translation* if

$$\forall (x, y, z) \in (\mathbb{R}^n)^3, \quad d(x + z, y + z) = d(x, y).$$

Definition 15. A *norm* over a vector space $\mathscr{E}$ is a function $N : \mathscr{E} \to \mathbb{R}^+$ such that
- $\forall x \in \mathscr{E},\ N(x) = 0 \Leftrightarrow x = o$ where o is the zero vector of $\mathscr{E}$.
- $\forall x \in \mathscr{E},\ \forall \lambda \in \mathbb{R},\ N(\lambda x) = |\lambda| N(x)$.
- $\forall (x, y) \in \mathscr{E}^2,\ N(x + y) \leqslant N(x) + N(y)$.

$(\mathscr{E}, N)$ is called *a normed vector space*.

Remark.
- If N is a norm over $\mathscr{E}$, then the function d_N such that: $\forall x, y \in \mathscr{E},\ d_N(x, y) = N(x - y)$ is a metric over $\mathscr{E}$. d_N is called the metric induced by the norm N.
- A metric induced by a norm is invariant under translation.

Example. $\mathscr{E} = \mathbb{R}^n$ and let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

$\forall p \geqslant 1,\ |x|_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$ and $|x|_\infty = \max(\{|x_i| \mid 1 \leqslant i \leqslant n\}) = \lim_{p \to \infty} |x|_p$ are norms over $\mathbb{R}^n$. The metrics induced by these norms are denoted d_p and d_∞, respectively.

Property 16 (Stoer and Witzgall [23]).
- *Let N be a norm on $\mathbb{R}^n$, and let $B_N = \{x \in \mathbb{R}^n \mid N(x) \leqslant 1\}$. Then B_N is a compact convex set with a non-empty interior and symmetrical relatively to the zero vector.*
- *Conversely, for every compact convex set $K \subset \mathbb{R}^n$ of dimension n, which is symmetrical relatively to the zero vector, there is precisely one norm N such that $B_N = K$. Moreover $\forall x \in \mathbb{R}^n,\ N(x) = \inf(\{r \in \mathbb{R}^+ \mid (1/r)x \in K\})$.*

2.3. Hausdorff metric

The definitions and results presented in this subsection can be found in [2, 8, 9].

Definition 17. Let $(\mathscr{E}, d)$ be a metric space, $\mathscr{H}(\mathscr{E})$ is the set of the non-empty compact subsets of $\mathscr{E}$.

On $\mathscr{H}(\mathscr{E})$, we will define a metric H_d, such that if $(\mathscr{E}, d)$ is a complete metric space then $(\mathscr{H}(\mathscr{E}), H_d)$ is a complete metric space.

(*A metric space $(\mathscr{E}, d)$ is complete if any Cauchy sequence in $(\mathscr{E}, d)$ has a limit point*).

Definition 18. Let $(\mathscr{E}, d)$ be a metric space and let $A \subset \mathscr{E}$ and $x_0 \in \mathscr{E}$; $d(x_0, A) = \inf(\{d(x_0, y) \mid y \in A\})$.

Definition 19. Let $(\mathscr{E}, d)$ be a metric space. Let $A, B \in \mathscr{H}(\mathscr{E})$. We define *the oriented Hausdorff metric* from a set $A \in \mathscr{H}(\mathscr{E})$ to a set $B \in \mathscr{H}(\mathscr{E})$ by $h_d(A, B) = \sup(\{d(a, B) \mid a \in A\})$.

Definition 20. Let $(\mathscr{E}, d)$ be a metric space. The *Hausdorff distance* between two non-empty compact sets $A, B \in \mathscr{H}(\mathscr{E})$ is defined by $H_d(A, B) = \max(h_d(A, B), h_d(B, A))$.

Definition 21. Let A,B be subsets of $\mathbb{R}^n$; the *Minkowski addition* of A and B is $A\oplus B=\{a+b\mid a\in A,\ b\in B\}=\bigcup_{a\in A}B(a)=\bigcup_{b\in B}A(b)$ where $\forall t\in\mathbb{R}^n$, $A(t)=A\oplus\{t\}$ is the *translation* of A by t.

Property 22. *Let d be a metric on $\mathbb{R}^n$ and let $A,B\in\mathscr{H}(\mathbb{R}^n)$, then $h_d(A,B)=\min(\{r\geqslant 0\mid A\subseteq\bigcup_{b\in B}\mathscr{B}_r^d(b)\})$ and thus*

$$H_d(A,B)=\min\left(\left\{r\geqslant 0\,\middle|\,A\subseteq\bigcup_{b\in B}\mathscr{B}_r^d(b)\ \text{and}\ B\subseteq\bigcup_{a\in A}\mathscr{B}_r^d(a)\right\}\right).$$

So, if the metric d is invariant under translation, then

$$\forall A,B\in\mathscr{H}(\mathbb{R}^n),\quad H_d(A,B)=\min(\{r\geqslant 0\mid A\subseteq B\oplus\mathscr{B}_r^d(o)\ \text{and}\ B\subseteq A\oplus\mathscr{B}_r^d(o)\}),$$

where o is the zero vector.

Property 23 (Barnsley [2], Goebel and Kirk [8], Hausdorff [9]). *Let $(\mathscr{E},d)$ be a metric space:*
- $\forall(A,B)\in\mathscr{H}(\mathscr{E})^2,\ H_d(A,B)=\sup(\{|d(x,A)-d(x,B)|\mid x\in\mathscr{E}\})$.
- $\forall(A,B,C,D)\in\mathscr{H}(\mathscr{E})^4,\ H_d(A\cup B,C\cup D)\leqslant\max(H_d(A,C),\ H_d(B,D))$.
- *$(\mathscr{E},d)$ is a complete metric space $\Leftrightarrow$ $(\mathscr{H}(\mathscr{E}),H_d)$ is a complete metric space.*

Remark.
- Let $\mathscr{F}'(\mathscr{E})$ be the set of non-empty closed sets of $\mathscr{E}$. Then, the functions h_d and H_d can be extended in a natural way as a function from $\mathscr{F}'(\mathscr{E})\times\mathscr{F}'(\mathscr{E})$ to $\mathbb{R}^+\cup\{+\infty\}$. H_d is a 'generalized metric' on $\mathscr{F}'(\mathscr{E})$ in the sense that it satisfies the axioms of a metric, but can take infinite values.
- Let d be a metric induced by a norm on $\mathscr{E}=\mathbb{R}^n$, $x\in\mathscr{E}$ and $F\in\mathscr{F}'(\mathscr{E})$, if $d(x,F)=r$, then there exists $y\in F$ such that $d(x,F)=d(x,y)$ because if $r'>r$ then $d(x,F)=d(x,F\cap\mathscr{B}_{r'}^d(x))$, so the compactness of $F\cap\mathscr{B}_{r'}^d(x)$ implies that there exists $y\in F\cap\mathscr{B}_{r'}^d(x)$ such that $d(x,F\cap\mathscr{B}_{r'}^d(x))=d(x,y)$. In particular, $\forall x\in\mathscr{E},\ \exists p\in\mathscr{D}_\rho$ such that $d(x,\mathscr{D}_\rho)=d(x,p)$ because $\mathscr{D}_\rho\in\mathscr{F}'(\mathscr{E})$.

2.4. Connectivity in the lattice $\rho\mathbb{Z}^2$

In this section the usual notions of 4-connexity and 8-connexity on $\mathbb{Z}^2$ are extended in a natural way to the lattice $\rho\mathbb{Z}^2$. These notions are in a certain sense the analogues in the discrete space of the Euclidean connectivity. Actually we prove, in Section 5 of this paper, that a Hausdorff discretization in $\rho\mathbb{Z}^2$ of a closed connected set F is 8-connected, and its maximal Hausdorff discretization is 4-connected.

Notation. Let $p\in\rho\mathbb{Z}^2$.
- $\mathscr{V}_4(p)=\{q\in\mathbb{Z}^2\mid d_1(p,q)=\rho\}$, $q\in\mathscr{V}_4(p)$ is called a 4-neighborhood of p;
- $\mathscr{V}_8(p)=\{q\in\mathbb{Z}^2\mid d_\infty(p,q)=\rho\}$, $q\in\mathscr{V}_8(p)$ is called a 8-neighborhood of p.

$\mathscr{V}_4(p)$ and $\mathscr{V}_8(p)$ are illustrated in Fig. 1.

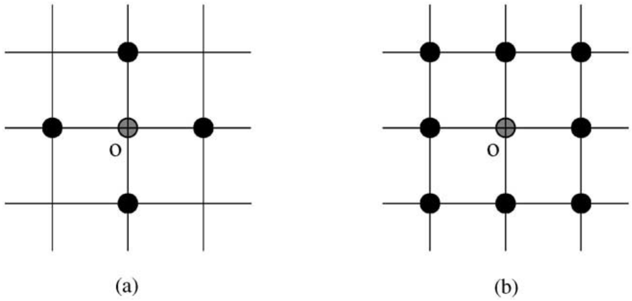


Fig. 1. The black points corresponds to (a) $\mathscr{V}_4(o)$, (b) $\mathscr{V}_8(o)$.

Definition 24. A subset S of $\rho\mathbb{Z}^2$ is *n-connected* for $n \in \{4, 8\}$, if for every $p, q \in S$ there exists a sequence $p_0, p_1, \ldots, p_l$ in S such that $p_0 = p$, $p_l = q$ and $p_{i+1} \in \mathscr{V}_n(p_i)$ for $i = 0, \ldots, l-1$. A subset S' of S is an *n-connected component* of S, if S' is a maximal (*relatively to the set inclusion*) n-connected subset of S.

3. Hausdorff discretization

In this section, we study a new framework of discretization of closed sets based on Hausdorff metric. First, we present the morphological discretization; actually, some classical discretizations, such as the supercover discretization are a special case of morphological discretization. There are relationships between some Hausdorff discretizations and morphological discretizations. In Section 3.2, we introduce a new framework of discretization based on the Hausdorff metric. In Section 3.3, we propose a new method for discretizing operators based on Hausdorff discretization.

Definition 25. Let $n \in \mathbb{N}^*$ and $\rho \in \mathbb{R}^+$, the *n-square lattice with step* ρ is the set $\mathscr{D}_\rho = \rho\mathbb{Z}^n$.

In all the following, we assume that we have as metric space $(\mathbb{R}^n, d)$, where d is a metric induced by a norm on $\mathbb{R}^n$, and as discrete space $\mathscr{D}_\rho = \rho\mathbb{Z}^n$, for a real number $\rho > 0$. So

$$\forall x \in \mathbb{R}^n, \quad \forall r \geqslant 0, \quad \mathscr{B}_r^d(x) \cap \mathscr{D}_\rho \textit{ is a finite set},$$

and for such distances, $M \subseteq \mathscr{D}_\rho \Rightarrow M \in \mathscr{F}(\mathbb{R}^n)$.

We refer, in the following, to the zero vector by o.

Definition 26. Let d be a metric on $\mathbb{R}^n$. The *covering radius* of the metric d is

$$r_c(\rho) = \sup(\{d(x, \mathscr{D}_\rho) \mid x \in \mathbb{R}^n\}).$$

So d induced by a norm implies that $r_c(\rho) = \rho r_c(1)$.

3.1. Morphological discretization

In this section we introduce the morphological discretization studied in [10–13]. Some Hausdorff discretizations can be viewed in a certain sense as morphological discretizations in which the structuring element is variable depending on the set to be discretized.

Definition 27. Let $X \subseteq \mathbb{R}^n$ and $\mathscr{S} \subseteq \mathbb{R}^n$, *the discretization by dilation of* X *by* $\mathscr{S}$ in $\mathscr{D}_\rho$ is the set $\Delta_\oplus^{\mathscr{S}}(X,\rho) = (X \oplus \check{\mathscr{S}}) \cap \mathscr{D}_\rho$ where $\check{\mathscr{S}} = \{-s \mid s \in \mathscr{S}\}$. $\mathscr{S}$ is called the *structuring element*.

Property 28. *Let* $X \subseteq \mathbb{R}^n$ *and let* $\mathscr{S} \subseteq \mathbb{R}^n$; *then* $\Delta_\oplus^{\mathscr{S}}(X,\rho) = \{p \in \mathscr{D}_\rho \mid X \cap \mathscr{S}(p) \neq \emptyset\}$, *where* $\mathscr{S}(p) = \mathscr{S} \oplus \{p\}$.

Remarks, definitions and notations. Several classical discretizations are particular cases of morphological discretization. Let $X \subseteq \mathbb{R}^n$, d be a metric induced by a norm, and let $\mathscr{S} \subseteq \mathbb{R}^n$:

- For $\mathscr{S} = \{o\}$, $\Delta_\oplus^{\mathscr{S}}(X,\rho)$ is *the discretization by sampling of* X.
- Let $r \in \mathbb{R}^+$. For $\mathscr{S} = \mathscr{B}_r^d(o)$, $\Delta_\oplus^{\mathscr{S}}(X,\rho)$ is called *the discretization of* X *of radius* r for the metric d and it is denoted in the following by $\Delta_r^d(X,\rho)$.
- For $r = r_c(\rho)$, $\Delta_r^d(X,\rho)$ is called *the discretization by overlapping of* X. For $d = d_\infty$ we have $r_c(\rho) = \rho/2$, so $\Delta_{\rho/2}^{d_\infty}(X,\rho) = \{p \in \mathscr{D}_\rho \mid \mathscr{C}(p,\rho) \cap X \neq \emptyset\}$ which is *the supercover discretization of* X where $\mathscr{C}(p,\rho)$ is the n-cube $\mathscr{B}_{\rho/2}^{d_\infty}(p) = \{x \in \mathbb{R}^n \mid d_\infty(x,p) \leqslant \rho/2\}$ [5].
  $\Delta_{\rho/2}^{d_\infty}(X,\rho)$ is denoted in the following by $\Delta_{\mathscr{SC}}(X,\rho)$.

But, there are other discretizations which are not in general particular cases of the morphological discretization:

Let $F \in \mathscr{F}'(\mathbb{R}^n)$ and $M \subseteq \mathscr{D}_\rho$. If $\forall p \in M$, $F \cap \mathscr{C}(p,\rho) \neq \emptyset$ and $F \subseteq \bigcup_{p \in M} \mathscr{C}(p,\rho)$ then M is called a *covering discretization* of E [1]. So the popular *supercover discretization* is the maximal *covering discretization*.

3.2. Characterization of Hausdorff discretization

Let F be a non-empty closed subset of $\mathbb{R}^n$, $S \subseteq \mathscr{D}_\rho$ is a Hausdorff discretization of F if it minimizes the Hausdorff distance to F. In this section, we study the properties of Hausdorff discretizations. In [19, 20, 26, 27], we have studied the Hausdorff discretization of compact sets.

Definition 29. Let $F \in \mathscr{F}'(\mathbb{R}^n)$.

- A set $S \subseteq \mathscr{D}_\rho$ is a *Hausdorff discretization* of F if $H_d(F,S) = \inf(\{H_d(F,S') \mid S' \subseteq \mathscr{D}_\rho\})$.
- $\mathscr{M}_H(F,\rho) = \{S \subseteq \mathscr{D}_\rho \mid H_d(F,S) = \inf(\{H_d(F,S') \mid S' \subseteq \mathscr{D}_\rho\})\}$ is the set of Hausdorff discretizations of F in $\mathscr{D}_\rho$.

- $\Delta_H(F,\rho)=(\bigcup_{S\in\mathscr{M}_H(F,\rho)} S)$ is called *the maximal Hausdorff discretization* of F.
- The value $r_H(F,\rho)=\sup(\{d(x,\mathscr{D}_\rho)\mid x\in F\})$ is called *the Hausdorff radius* of the closed set F for the metric d in the discrete space $\mathscr{D}_\rho$.

We will now characterize the Hausdorff discretization.

Theorem 1. *Let $F\in\mathscr{F}'(\mathbb{R}^n)$; then*
- *$\mathscr{M}_H(F,\rho)$ is non-void and if $S\in\mathscr{M}_H(F,\rho)$ then $H_d(F,S)=r_H(F,\rho)$,*
- *$\Delta_H(F,\rho)=\{p\in\mathscr{D}_\rho\mid d(p,F)\leqslant r_H(F,\rho)\}\in\mathscr{M}_H(F,\rho)$,*
- *if $(S_i)_{i\in I}$ is a family of members of $\mathscr{M}_H(F,\rho)$, then $\bigcup_{i\in I} S_i\in\mathscr{M}_H(F,\rho)$,*
- *if $(S_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathscr{M}_H(F,\rho)$ (relatively to the set inclusion) then $\bigcap_{n\in\mathbb{N}} S_n\in\mathscr{M}_H(F,\rho)$ and*
- *$r_H(F,\rho)\leqslant r_c(\rho)$.*

Proof.
- Let $M=\{p\in\mathscr{D}_\rho\mid d(p,F)\leqslant r_H(F,\rho)\}$. $M\neq\emptyset$ because if $x\in F$ and $r>r_H(F,\rho)$ then $\mathscr{B}_r^d(x)\cap\mathscr{D}_\rho$ is a non-void finite set and for all $q\in(\mathscr{D}_\rho\setminus\mathscr{B}_r^d(x))$, $d(x,q)>r$, so there exists $p\in\mathscr{B}_r^d(x)\cap\mathscr{D}_\rho$ such that $d(p,x)=d(x,\mathscr{D}_\rho)\leqslant r_H(F,\rho)$ and thus $M\neq\emptyset$.
  We will prove that $M\in\mathscr{M}_H(F,\rho)$. $\forall p\in M$, $d(p,F)\leqslant r_H(F,\rho)$, so $h_d(M,F)\leqslant r_H(F,\rho)$.
  For all $x\in F$, there exists $p\in\mathscr{D}_\rho$ such that $d(x,p)=d(x,\mathscr{D}_\rho)\leqslant r_H(F,\rho)$. So $p\in M$, thus $h_d(F,M)\leqslant r_H(F,\rho)$. Therefore $H_d(F,M)\leqslant r_H(F,\rho)$.
  Let $S\in\mathscr{P}(\mathscr{D}_\rho)$ and assume that $H_d(F,S)<r_H(F,\rho)$, so $r_H(F,\rho)=\sup(\{d(x,\mathscr{D}_\rho)\mid x\in F\})\leqslant h_d(F,S)<r_H(F,\rho)$, which is absurd. So $H_d(F,M)=r_H(F,\rho)$. Thus $M\in\mathscr{M}_H(F,\rho)$, $\mathscr{M}_H(F,\rho)\neq\emptyset$ and if $S\in\mathscr{M}_H(F,\rho)$ then $H_d(F,S)=r_H(F,\rho)$.
- Let $S\in\mathscr{M}_H(F,\rho)$, then $\forall p\in S$, $\exists x\in F$ such that $d(x,p)\leqslant r_H(F,\rho)$. So $S\subseteq M=\{p\in\mathscr{D}_\rho\mid d(p,F)\leqslant r_H(F,\rho)\}$. Thus $\Delta_H(F,\rho)=\{p\in\mathscr{D}_\rho\mid d(p,F)\leqslant r_H(F,\rho)\}$.
- Let $(S_i)_{i\in I}$ be a family of members of $\mathscr{M}_H(F,\rho)$. So if $p\in\bigcup_{i\in I} S_i$ then $\exists i\in I$ such that $p\in S_i$ and thus there exists x in F such that $d(x,p)\leqslant r_H(F,\rho)$, so $h_d(\bigcup_{i\in I} S_i,F)\leqslant r_H(F,\rho)$.
  If $x\in F$ then there exists $p\in S_{i_0}$ such that $d(x,p)\leqslant r_H(F,\rho)$, so $h_d(F,\bigcup_{i\in I} S_i)\leqslant r_H(F,\rho)$, so $\bigcup_{i\in I} S_i\in\mathscr{M}_H(F,\rho)$.
- Let $x\in F$, then $V(x)=\mathscr{B}^d_{r_H(F,\rho)}(x)\cap\mathscr{D}_\rho$ is non-empty finite set of $\mathscr{D}_\rho$. Put $V_n(x)=V(x)\cap S_n$. So $V_0(x)$ is a finite set, $(V_n(x))_{n\in\mathbb{N}}$ is a decreasing sequence and $V_n(x)\neq\emptyset$ for all $n\in\mathbb{N}$. Thus there exists $m\in\mathbb{N}$, such that $V_n(x)=V_m(x)$ for all $n\geqslant m$. Therefore $V_m(x)\subseteq\bigcap_{n\in\mathbb{N}} S_n$. Thus $\bigcap_{n\in\mathbb{N}} S_n\neq\emptyset$ and there exists $p\in\bigcap_{n\in\mathbb{N}} S_n$ such that $d(x,p)\leqslant r_H(F,\rho)$. So $h_d(F,\bigcap_{n\in\mathbb{N}} S_n)\leqslant r_H(F,\rho)$.
  If $p\in\bigcap_{n\in\mathbb{N}} S_n$ then $p\in S_0$, thus there exists $x\in F$ such that $d(x,p)\leqslant r_H(F,\rho)$, so $h_d(\bigcap_{n\in\mathbb{N}} S_n,F)\leqslant r_H(F,\rho)$, so $\bigcap_{n\in\mathbb{N}} S_n\in\mathscr{M}_H(F,\rho)$.
- $r_H(F,\rho)=\sup(\{d(x,\mathscr{D}_\rho)\mid x\in F\})\leqslant\sup(\{d(x,\mathscr{D}_\rho)\mid x\in\mathbb{R}^n\})=r_c(\rho)$. $\square$

Remark. In [19, 26, 27] we have proved that, if $K\in\mathscr{H}(\mathbb{R}^n)$, then $\mathscr{M}_H(K,\rho)$ is finite and $\forall S\in\mathscr{M}_H(F,\rho)$, S is finite. Actually, if $r=\sup(\{d(o,x)\mid x\in K\})+r_H(K,\rho)$ where

o is the zero vector (*the fact that K is a compact set implies that r is finite*), then $\Delta_H(K,\rho) \subseteq (\mathscr{B}_r^d(o) \cap \mathscr{D}_\rho)$, which is a finite set.

Property 30. *Let $F \in \mathscr{F}'(\mathbb{R}^n)$, $r \in \mathbb{R}^+$ and let $S \subseteq \mathscr{D}_\rho$ such that $F \subseteq \bigcup_{p \in S} \mathscr{B}_r^d(p)$ and $\forall p \in S$, $\mathscr{B}_r^d(p) \cap F \neq \emptyset$. Then $H_d(F,S) \leqslant r$. So if $r = r_H(F,\rho)$ then $S \in \mathscr{M}_H(F,\rho)$.*

Proof. $F \subseteq \bigcup_{p \in S} \mathscr{B}_r^d(p)$ implies that $h_d(F,S) \leqslant r$. Let $p \in S$ then $\mathscr{B}_r^d(p) \cap F \neq \emptyset$, so there exists $x \in F$ such that $d(x,p) \leqslant r$, so $p \in \mathscr{B}_r^d(x)$. Thus, $h_d(S,F) \leqslant r$ and hence $H_d(F,S) \leqslant r$. □

Remark. Let $F \in \mathscr{F}'(\mathbb{R}^n)$ and put $r = r_H(F,\rho)$.
- $\Delta_H(F,\rho) = \Delta_r^d(F,\rho)$. So the maximal Hausdorff discretization of F is in a certain sense a morphological discretization with a structuring element $\mathscr{S} = \mathscr{B}^d_{r_H(F,\rho)}(o)$ depending on F.
- $S \in \mathscr{M}_H(F,\rho)$ iff $F \subseteq \bigcup_{p \in S} \mathscr{B}_r^d(p)$ and $\forall p \in S$, $\mathscr{B}_r^d(p) \cap F \neq \emptyset$. So a Hausdorff discretization S of F is similar to a covering discretization S' of F if we replace for $p \in S$ the ball $\mathscr{B}^d_{r_H(F,\rho)}(p)$, which depends on F, by the square $\mathscr{C}(p',\rho)$ for $p' \in S'$.

In Fig. 2 we illustrate the construction of Hausdorff discretizations for a closed set F: computing the Hausdorff radius (maximal distance from points of F to the discrete space), one takes for $\Delta_H(F,\rho)$ all discrete points p such that the ball of center p and Hausdorff radius intersects F; any subset M of $\Delta_H(F,\rho)$ such that the corresponding balls for $p \in M$ cover F, will be a Hausdorff discretization.

In the following proposition, we prove that 'digital geometry' converges relatively to the Hausdorff metric to 'Euclidean geometry', as in [6], by using lattices with decreasing resolution.

Proposition 2. *Let $F \in \mathscr{F}'(\mathbb{R}^n)$, then for any choice of $M^\rho \in \mathscr{M}_H(F,\rho)$,*

$$\lim_{\rho \to 0} H_d(F, M^\rho) = 0.$$

Proof. $H_d(F,M^\rho) = r_H(F,\rho) \leqslant r_c(\rho)$. So $r_c(\rho) = \rho r_c(1)$ implies that $\lim_{\rho \to 0} r_c(\rho) = 0$, thus $\lim_{\rho \to 0} H_d(F,M^\rho) = 0$. □

Remark. Let $BH_d : \mathscr{F}'(\mathbb{R}^n) \times \mathscr{F}'(\mathbb{R}^n) \to \mathbb{R}^+$ defined by

$$BH_d(F,F') = \sup(\{|d(x,F) - d(x,F')|\mathrm{e}^{-d(O,x)} | x \in \mathbb{R}^n\}) \quad \textit{for all } F,F' \in \mathscr{F}'(\mathbb{R}^n),$$

where O is any fixed point in $\mathbb{R}^n$ for example the zero vector.

Then BH_d is a metric on $\mathscr{F}'(\mathbb{R}^n)$; BH_d is called the Busemann–Hausdorff metric [4, 12, 17, 22]. The topology induced by BH_d is the hit-or-miss topology if for example every ball (*relatively to the metric d*) is a compact set [12].

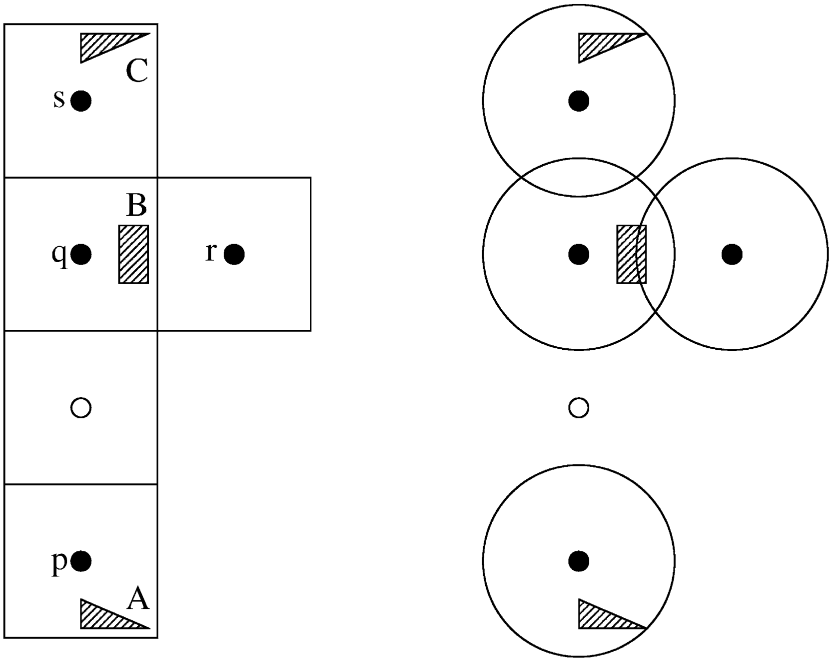


Fig. 2. Left: The set $F = A \cup B \cup C$ overlayed with discrete points p, q, r, s and their square cells $\mathscr{C}(p,1), \mathscr{C}(q,1), \mathscr{C}(r,1), \mathscr{C}(s,1)$. Right: For $d = d_2$ (the Euclidean distance), the maximal Hausdorff discretization of F is $\{p,q,r,s\}$; indeed, we show the circles of radius $r_H(F,\rho)$ centered about these points. The unique other Hausdorff discretization of F is $\{p,q,s\}$. So $\mathscr{M}_H(F,1) = \{\{p,q,s\},\{p,q,r,s\}\}$.

By Property 23, we have $\forall F,F' \in \mathscr{F}'(\mathbb{R}^n)$, $BH_d(F,F') \leqslant H_d(F,F')$ and $BH_d(F,F')$ is always finite. So for any choice of $M^\rho \in \mathscr{M}_H(F,\rho)$, $\lim_{\rho\to 0} BH_d(F,M^\rho) = 0$, which implies that a Hausdorff discretization M^ρ of the closed set F converges to F for the hit-or-miss topology when the resolution ρ of the discrete space converges to 0.

In several papers and books [10–12, 22], the authors do not prove the convergence of a discretization M^ρ to F, but the convergence of the 'reconstruction' of the discretization of F to F if the resolution ρ of the discrete space converges to 0.

3.3. Discretization of operators

Based on Hausdorff discretization of closed sets, we propose in this section a framework of discretization of some classes of operators on closed sets.

Notation. Let E be a set:
- $\mathscr{P}'(E)$ is the set of non-empty subsets of E.
- $\mathscr{P}_f(E)$ is the set of non-empty finite subsets of E.

Definition 31. Let $\rho > 0$. The Hausdorff choice discretization in the discrete space $\mathscr{D}_\rho$ is the set $\mathscr{S}_H^\rho$ of functions σ from $\mathscr{F}'(\mathbb{R}^n)$ to $\mathscr{P}(\mathscr{D}_\rho)$ such that $\sigma(F) \in \mathscr{M}_H(F, \rho)$ for all $F \in \mathscr{F}'(\mathbb{R}^n)$ (σ is a 'choice' function).

Definition 32. Let Φ be an operator on $\mathscr{F}'(\mathbb{R}^n)$, a function Φ^ρ on $\mathscr{P}'(\mathscr{D}_\rho)$ is a Hausdorff discretization of Φ, if there exists $\sigma \in \mathscr{S}_H^\rho$ such that for every $E \in \mathscr{P}'(\mathscr{D}_\rho)$, $\Phi_\rho(E) = \sigma(\Phi(E))$. In the following we denote Φ^ρ by Φ_σ because it only depends on Φ and σ.

Theorem 3. *Let Φ be a continuous operator on $\mathscr{K}(\mathbb{R}^n)$ relatively to the Hausdorff metric H_d, then $\forall \sigma_1^\rho, \sigma_2^\rho \in \mathscr{S}_H^\rho$, $\forall K \in \mathscr{K}(\mathbb{R}^n)$, $\lim_{\rho \to 0} \Phi_{\sigma_1^\rho}(\sigma_2^\rho(K)) = \Phi(K)$.*

Proof. Let $\varepsilon > 0$. Then $\lim_{\rho \to 0} r_c(\rho) = 0$ implies that $\exists \rho'$ such that $r_c(\rho) \leqslant \varepsilon/2$, for all $\rho \leqslant \rho'$. So $H_d(\sigma_1^\rho(\Phi(\sigma_2^\rho(K))), \Phi(\sigma_2^\rho(K))) \leqslant r_c(\rho) \leqslant \varepsilon/2$ for all $\rho \leqslant \rho'$.

Φ is continuous and $\lim_{\rho \to 0} H_d(\sigma_2^\rho(K), K)) = 0$ imply that $\exists \rho''$ such that $H_d(\Phi(\sigma_2^\rho(K)), \Phi(K)) \leqslant \varepsilon/2$ for all $\rho \leqslant \rho''$. So $H_d(\sigma_1^\rho(\Phi(\sigma_2^\rho(K))), \Phi(K)) \leqslant H_d(\sigma_1^\rho(\Phi(\sigma_2^\rho(K))), \Phi(\sigma_2^\rho(K))) + H_d(\Phi(\sigma_2^\rho(K)), \Phi(K)) \leqslant \varepsilon$ for all $\rho \leqslant \min(\rho', \rho'')$. $\square$

The following diagram illustrates the proposed scheme of discretization of some classes of operators on compact sets.

$$\begin{tikzcd}
\mathscr{K}(\mathbb{R}^n) \arrow[r, "\Phi"] \arrow[d, "\sigma_2^\rho"'] & \mathscr{K}(\mathbb{R}^n) \arrow[d, "\sigma_1^\rho"] \\
\mathscr{P}'(\mathscr{D}_\rho) \arrow[ur, "\Phi"] \arrow[r, "\Phi_{\sigma_1}"'] & \mathscr{P}'(\mathscr{D}_\rho)
\end{tikzcd}$$

Property 33. *$\oplus$ is a continuous function from $\mathscr{K}(\mathbb{R}^n) \times \mathscr{K}(\mathbb{R}^n)$ to $\mathscr{K}(\mathbb{R}^n)$.*

Proof. Let $X, X', Y, Y' \in \mathscr{K}(\mathbb{R}^n)$. $h_d(X, X') \leqslant \varepsilon_1$ and $h_d(Y, Y') \leqslant \varepsilon_2$ implies that $X \subseteq X' \oplus \mathscr{B}_{\varepsilon_1}^d(o)$ and $Y \subseteq Y' \oplus \mathscr{B}_{\varepsilon_2}^d(o)$, so $X \oplus Y \subseteq X' \oplus Y' \oplus \mathscr{B}_{\varepsilon_1}^d(o) \oplus \mathscr{B}_{\varepsilon_2}^d(o) \subseteq X' \oplus Y' \oplus \mathscr{B}_{\varepsilon_1+\varepsilon_2}^d(o)$. Thus $h_d(X \oplus X', Y \oplus Y') \leqslant \varepsilon_1 + \varepsilon_2$. So $H_d(X \oplus Y, X' \oplus Y') \leqslant H_d(X, X') + H_d(Y, Y')$. $\square$

Corollary 4. *Let $K \in \mathscr{K}(\mathbb{R}^n)$ and let the function δ_K on $\mathscr{K}(\mathbb{R}^n)$ such that for all $K' \in \mathscr{K}(\mathbb{R}^n)$, $\delta_K(K') = K' \oplus K$. Then, δ_K is continuous. δ_K is called the* dilation *by K.*

The last corollary implies that if δ_K^ρ is a Hausdorff discretization of dilation δ_K in $\mathscr{D}_\rho$, then for all compact sets K' for any choice of $S^\rho \in \mathscr{M}_H(K', \rho)$, $\lim_{\rho \to 0} \delta_K^\rho(S^\rho) = \delta_K(K') = K \oplus K'$.

In the following proposition we prove that some of the Hausdorff discretizations of a dilation by compact set can be expressed as a dilation by a finite subset of the discrete space.

Proposition 5. *Let $K \in \mathscr{H}(\mathbb{R}^n)$, and let $M^\rho \in \mathscr{M}_H(K, \rho)$, then the function δ_{M^ρ} on $\mathscr{P}_f(\mathscr{D}_\rho)$, defined by $\delta_{M^\rho}(E) = E \oplus M^\rho$ for every $E \in \mathscr{P}_f(\mathscr{D}_\rho)$, is a Hausdorff discretization of δ_K.*

Proof. Let $E \in \mathscr{P}_f(\mathscr{D}_\rho)$, it is sufficient to prove that $(E \oplus M^\rho) \in \mathscr{M}_H(E \oplus K, \rho)$ for every $E \in \mathscr{P}_f(\mathscr{D}_\rho)$.

Let $x \in E \oplus K$, then there exists $e \in E$, $k \in K$ such that $x = e + k$ and as d is invariant under translation and $e \in \mathscr{D}_\rho$, $d(x, \mathscr{D}_\rho) = d(x - e, \mathscr{D}_\rho - e) = d(k, \mathscr{D}_\rho)$. So $r_H(E \oplus K, \rho) = r_H(K, \rho)$.

We also have $E \oplus K \subseteq E \oplus (M^\rho \oplus \mathscr{B}^d_{r_H(K,\rho)}(o))$ and $E \oplus M^\rho \subseteq E \oplus (K \oplus \mathscr{B}^d_{r_H(K,\rho)}(o))$. So $H_d(E \oplus K, E \oplus M^\rho) = r_H(E \oplus K, \rho)$. Thus $E \oplus M^\rho \in \mathscr{M}_H(E \oplus K, \rho)$. $\square$

Definition 34. Let $\Phi : \mathscr{P}(\mathscr{E}) \to \mathscr{P}(\mathscr{E})$. Φ is called a *point operator* if there exists a function $f : \mathscr{E} \to \mathscr{E}$ such that $\Phi(E) = \{f(x) \mid x \in E\}$ for $E \subseteq \mathscr{E}$, Φ is called the *point operator corresponding to* the function f. In the following, Φ is denoted by Φ_f for referencing f.

Property 35. *Let f be a continuous function on $\mathbb{R}^n$ relatively to a metric d, then the point operator Φ_f corresponding to f on $\mathscr{H}(\mathbb{R}^n)$ defined by $\forall K \in \mathscr{H}(\mathbb{R}^n)$, $\Phi_f(K) = \{f(x) \mid x \in K\}$ is a continuous function relatively to H_d.*

Proof. Let $K \in \mathscr{H}(\mathbb{R}^n)$ and $r > 0$. Then the dilation of radius r of the compact set K, $K_r = \delta_r(K) = \bigcup_{x \in K} \mathscr{B}^d_r(x)$, is a compact set because K_r is closed and bounded. So f is uniformly continuous on K_r. So for every $\varepsilon > 0$ there exists $\eta > 0$ such that for all $x, y \in K_r$, $d(x, y) < \eta$ implies that $d(f(x), f(y)) < \varepsilon$ (*we can assume* $\eta < r$). Then, for $K' \in \mathscr{H}(\mathbb{R}^n)$ with $H(K, K') < \eta$, we have K' is a subset of K_r. For all $x \in K'$, $d(x, K) < \eta$, therefore there exists $y \in K$ with $d(x, y) < \eta$, then $d(f(x), f(y)) < \varepsilon$, and so $h_d(f(K'), f(K)) \leqslant \varepsilon$.

In the same way, if $x \in K$ then $d(x, K') < \eta$, therefore there exists $y \in K'$ with $d(x, y) < \eta$, which implies that $d(f(x), f(y)) < \varepsilon$.

So $H_d(f(K), f(K') \leqslant \varepsilon$. $\square$

Remark.

- The last properties imply that if Φ_f is the point operator corresponding to a continuous function f and Φ_f^ρ is a Hausdorff discretization of Φ_f in $\mathscr{D}_\rho$, then for all compact sets K, for any choice of $S^\rho \in \mathscr{M}_H(K, \rho)$, $\lim_{\rho \to 0} \Phi_f^\rho(S^\rho) = \Phi_f(K)$.
- There are other alternatives for defining a Hausdorff discretization of a point operator corresponding to a continuous function f on $\mathbb{R}^n$; it can be defined by using Hausdorff

discretizations of the graph $Gr(f)$ of f ($Gr(f)=\{(x,f(x))\,|\,x\in\mathbb{R}^n\}$ *is a closed set of* $\mathbb{R}^{2n}$: $Gr(f)\in\mathscr{F}'(\mathbb{R}^{2n})$).

4. Homogeneous metric and Hausdorff discretization

In this section we introduce a new notion: *a homogeneous metric*. We present some properties of a homogeneous metric, we refine the characterization of Hausdorff discretizations for homogeneous metric and we compare Hausdorff discretizations to other discretization schemes. Actually, we study the relationship between Hausdorff discretizations and covering discretizations, and we compare the Bresenham discretizations [3] to Hausdorff discretizations for straight lines of $\mathbb{R}^2$.

Definition 36. A metric d over $\mathbb{R}^n$ is called *cellular* if

$$\forall x\in\mathbb{R}^n,\quad \forall p,q\in\mathbb{Z}^n,\quad x\in\mathscr{C}(p,1)\Rightarrow d(p,x)\leqslant d(q,x).$$

In particular, if $x\in\mathscr{C}(p,1)\cap\mathscr{C}(q,1)$, then $d(p,x)=d(q,x)$.

All the usual metrics are cellular: d_p is cellular for all $p\geqslant 1$ and for $p=\infty$.
In Fig. 3, we give an example of metric induced by a norm which is not cellular.

Definition 37. A norm N on $\mathbb{R}^n$ is *homogeneous* if $\forall(x_1,\dots,x_n)\in\mathbb{R}^n$, $\forall(\varepsilon_1,\dots,\varepsilon_n)\in\{-1,1\}^n$, for every permutation σ of $\{1,\dots,n\}$, $N(\varepsilon_1 x_{\sigma(1)},\dots,\varepsilon_n x_{\sigma(n)})=N(x_1,\dots,x_n)$. So, if $n=2$, then N is *homogeneous* iff $\forall(x_1,x_2)\in\mathbb{R}^2$, $N(x_1,x_2)=N(-x_1,x_2)=N(x_2,x_1)$.
A metric induced by a homogeneous norm is called a *homogeneous* metric.

Theorem 6. *Let d be a homogeneous metric induced by the norm N; then*
- *d is cellular,*
- *$r_{\mathrm{c}}(1)=\frac{1}{2}N(1,\dots,1)$ and*
- *$\mathscr{B}^{d_\infty}_{1/2}(o)\subseteq\mathscr{B}^{d}_{r_{\mathrm{c}}(1)}(o)\subseteq\mathscr{B}^{d_1}_{n/2}(o)$.*

Proof.
- Let $(x_1,\dots,x_n),(y_1,\dots,y_n)\in\mathbb{R}^n$; if $|x_i|\leqslant|y_i|$ for $i=1,\dots,n$, then $(x_1,\dots,x_n)$ is in the convex hull of $\{(\varepsilon_1 y_1,\dots,\varepsilon_n y_n)\,|\,(\varepsilon_1,\dots,\varepsilon_n)\in\{-1,1\}^n\}=\{P_i\,|\,i\in I\}$.
  So there exists $\lambda_i\geqslant 0$, for $i\in I$ such that $\sum_{i\in I}\lambda_i=1$ and $(x_1,\dots,x_n)=\sum_{i\in I}\lambda_i P_i$. So $N(x_1,\dots,x_n)\leqslant\sum_{i\in I}\lambda_i N(P_i)=N(y_1,\dots,y_n)$ because N is a homogeneous norm.
  Let $P=(p_1,\dots,p_n)$, $Q=(q_1,\dots,q_n)\in\mathbb{Z}^n$ and let $X=(x_1,\dots,x_n)\in\mathscr{C}(P,1)$. $|x_i-p_i|\leqslant\frac{1}{2}$, so $|x_i-p_i|\leqslant|x_i-q_i|$ for $i=1,\dots,n$. Thus $d(X,P)=N(X-P)\leqslant N(X-Q)=d(X,Q)$. So d is a cellular metric.
- d is invariant under translation, so $r_{\mathrm{c}}(1)=\sup(\{d(x,\mathbb{Z}^n)\,|\,x\in\mathbb{R}^n\})=\sup(\{d(x,\mathbb{Z}^n)\,|\,x\in\mathscr{C}(o,1)\})=\sup(\{d(o,x)\,|\,x\in\mathscr{C}(o,1)\})$ because d is cellular. Let $x\in\mathscr{C}(o,1)$, then x is in the convex hull of $\{\frac{1}{2}(\varepsilon_1,\dots,\varepsilon_n)\,|\,(\varepsilon_1,\dots,\varepsilon_n)\in\{-1,1\}^n\}=\{P_1,\dots,P_{2^n}\}$.

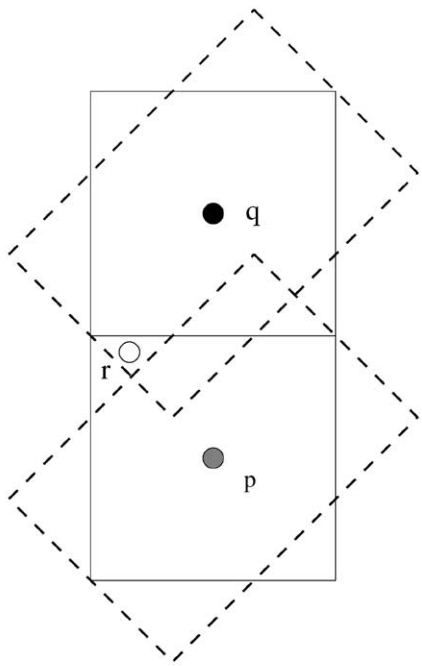


Fig. 3. The metric d induced by a norm such that $\mathscr{B}_1^d(p)$ is dashed is not cellular because $r \in \mathscr{C}(p,1)$ and $d(q,r) < d(p,r)$.

So there exists $\lambda_1, \ldots, \lambda_{2^n} \geqslant 0$ such that $\sum_{i=1}^{2^n} \lambda_i = 1$ and $x = \sum_{i=1}^{2^n} \lambda_i P_i$. Thus $N(x) \leqslant \sum_{i=1}^{2^n} \lambda_i N(P_i) = \frac{1}{2} N(1, \ldots, 1)$ because N is a homogeneous norm. Therefore, $r_c(1) = \frac{1}{2} N(1, \ldots, 1)$ and $\mathscr{C}(o,1) = \mathscr{B}_{1/2}^{d_\infty}(o) \subseteq \mathscr{B}_{r_c(1)}^d(o)$.

- Let $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and S_n be the set of all permutations of $\{1, \ldots, n\}$, then $\sum_{\sigma \in S_n} N(|x_{\sigma(1)}|, \ldots, |x_{\sigma(n)}|) \geqslant N(\sum_{\sigma \in S_n}(|x_{\sigma(1)}|, \ldots, |x_{\sigma(n)}|)) = (n-1)!(|x_1| + \cdots + |x_n|) N(1, \ldots, 1)$. So $(n-1)!(|x_1| + \cdots + |x_n|) N(1, \ldots, 1) \leqslant n! N(x_1, \ldots, x_n)$. Thus $(|x_1| + \cdots + |x_n|) 2 r_c(1) \leqslant n N(x_1, \ldots, x_n)$. So $\mathscr{B}_{r_c(1)}^d(o) \subseteq \mathscr{B}_{n/2}^{d_1}(o)$. $\square$

Example. In $\mathbb{R}^n$, $\forall p \geqslant 1$, d_p is a homogeneous metric, and thus $r_c(1) = (n^{1/p}/2)$; d_∞ is also a homogeneous metric then $r_c(1) = \frac{1}{2}$.

Remark. If d is a homogeneous metric on $\mathbb{R}^n$, then

- $\mathbb{R}^n = \bigcup_{p \in \mathscr{D}_\rho} \mathscr{B}_{r_c(\rho)}^d(p)$.
- $\forall x \in \mathbb{R}^n$, $\mathscr{B}_{1/2}^{d_\infty}(x) \subseteq \mathscr{B}_{r_c(1)}^d(x) \subseteq \mathscr{B}_{n/2}^{d_1}(x)$.
- If $n = 2$, then $\forall r > 0$, $\mathscr{B}_{r/2r_c(1)}^{d_\infty}(o) \subseteq \mathscr{B}_r^d(o) \subseteq \mathscr{B}_{r/r_c(1)}^{d_1}(o)$. This inclusion is illustrated for the metric d_2 in Fig. 4.

Property 38. *Let d be a homogeneous metric on $\mathbb{R}^2$ and $p, q \in \rho\mathbb{Z}^2$ such that $p \neq q$, and let $r \in \mathbb{R}^+$.*

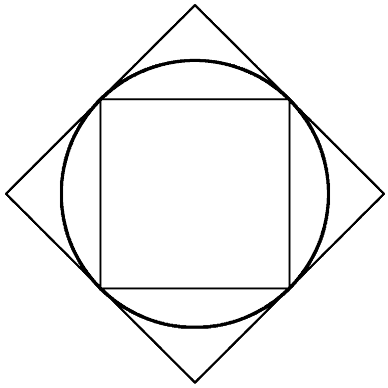


Fig. 4. Illustration in $\mathbb{R}^2$ of the inclusion $\mathscr{B}^{d_\infty}_{1/2}(o) \subseteq \mathscr{B}^{d}_{r_c(1)}(o) \subseteq \mathscr{B}^{d_1}_{1}(o)$ for $d = d_2$.

(i) *If* $r < r_c(\rho)$, *then* $(\mathscr{B}^d_r(p) \cap \mathscr{B}^d_r(q) \neq \emptyset \Rightarrow q \in \mathscr{V}_4(p))$.
(ii) *If* $r = r_c(\rho)$, *then* $(\mathscr{B}^d_r(p) \cap \mathscr{B}^d_r(q) \neq \emptyset \Rightarrow q \in \mathscr{V}_8(p))$.

Proof. (i) $\mathscr{B}^d_r(p) \cap \mathscr{B}^d_r(q) \subseteq \mathscr{B}^{d_1}_{r/r_{c(1)}}(p) \cap \mathscr{B}^{d_1}_{r/r_{c(1)}}(q)$. So if $\mathscr{B}^{d_1}_{r/r_{c(\rho)}}(p) \cap \mathscr{B}^{d_1}_{r/r_{c(\rho)}}(q) \neq \emptyset$ then $q \in \mathscr{V}_4(p)$ because $(r/r_{c(1)}) < \rho$.

(ii) $\mathscr{B}^d_{r_{c(\rho)}}(p) \cap \mathscr{B}^d_{r_{c(\rho)}}(q) \subseteq \mathscr{B}^{d_1}_{\rho}(p) \cap \mathscr{B}^{d_1}_{\rho}(q)$. So if $\mathscr{B}^{d_1}_{\rho}(p) \cap \mathscr{B}^{d_1}_{\rho}(q) \neq \emptyset$ then $q \in \mathscr{V}_8(p)$. □

Remark.
- If d is a homogeneous metric on $\mathbb{R}^2$ and $p, q \in \rho\mathbb{Z}^2$ such that $p \neq q$, then:
  - If $\mathscr{C}(p,\rho) \cap \mathscr{B}^d_{r_{c(\rho)}}(q) \neq \emptyset$, then $q \in \mathscr{V}_8(p)$. If $q \notin (\mathscr{V}_8(p) \cup \{p\})$ then $\mathscr{B}^d_{r_{c(\rho)}}(p) \cap \mathscr{B}^d_{r_{c(\rho)}}(q) = \emptyset$.
  - If $r < r_c(\rho)$ and $\mathscr{C}(p,\rho) \cap \mathscr{B}^d_r(q) \neq \emptyset$, then $q \in \mathscr{V}_4(p)$, more precisely, if $\mathscr{C}(p,\rho) \cap int(\mathscr{B}^d_{r_{c(\rho)}}(q)) \neq \emptyset$ then $q \in \mathscr{V}_4(p)$.
- More generally, if d is a homogeneous metric in $\mathbb{R}^n$ and $p, q \in \rho\mathbb{Z}^n$ such that $p \neq q$, then
  - If $d_1(p,q) > \rho n$, then $\mathscr{B}^d_{r_c(\rho)}(p) \cap \mathscr{B}^d_{r_c(\rho)}(q) = \emptyset$.
  - If $d_1(p,q) = \rho n$, then $\mathscr{B}^d_{r_c(\rho)}(p) \cap \mathscr{B}^d_{r_c(\rho)}(q) \cap \mathscr{C}(p,\rho) = \mathscr{C}(p,\rho) \cap \mathscr{C}(q,\rho)$, which is non-empty iff p and q are diagonally adjacent, except for $d = \alpha d_1$.

Definition 39. Let d be a metric on $\mathbb{R}^n$ and let $p \in \mathbb{R}^n$ and $r \in \mathbb{R}^+$,

$$C^d_r(p) = \{x \in \mathbb{R}^n \mid d(x,p) = r\}.$$

$C^d_r(p)$ is the circle of radius r and center p relatively to the metric d.

Property 40. *Let d be a homogeneous metric on $\mathbb{R}^2$ induced by a norm N, then*
(i) $\frac{1}{2}\{-1,1\}^2 \subseteq (C^d_{r_c(1)}(o) \cap C^{d_\infty}_{1/2}(o))$.

(ii) $\frac{1}{2}\{-1,1\}^2 \subset (C^d_{r_c(1)}(o) \cap C^{d_\infty}_{1/2}(o)) \Leftrightarrow \exists \alpha \in \mathbb{R}^+$ *such that* $d = \alpha d_\infty$.
(iii) *If* $K \in \mathscr{H}(\mathbb{R}^2)$ *and* $d \neq \alpha d_\infty$, *then* $(r_H(K,1) = r_c(1) \Leftrightarrow K \cap (\{(\frac{1}{2},\frac{1}{2})\} \oplus \mathbb{Z}^2) \neq \emptyset)$.

Proof. (i) $r_c(1) = \frac{1}{2}N(1,1) = \frac{1}{2}N(\varepsilon_1,\varepsilon_2)$ for all $(\varepsilon_1,\varepsilon_2) \in \{-1,1\}^2$ because N is homogeneous. So $\frac{1}{2}\{-1,1\}^2 \subseteq (C^d_{r_c(1)}(o) \cap C^{d_\infty}_{1/2}(o))$.

(ii) ($\Rightarrow$) If $\frac{1}{2}\{-1,1\}^2 \subset (C^d_{r_c(1)}(o) \cap C^{d_\infty}_{1/2}(o))$ then there exists $-\frac{1}{2} < x < \frac{1}{2}$ such that $N(x,\frac{1}{2}) = r_c(1)$. Let $x < y \leqslant \frac{1}{2}$. Then there exists $0 \leqslant \lambda \leqslant 1$ such that $(x,\frac{1}{2}) = \lambda(-\frac{1}{2},\frac{1}{2}) + (1-\lambda)(y,\frac{1}{2})$. So $N(y,\frac{1}{2}) \leqslant r_c(1)$ and $N(x,\frac{1}{2}) \leqslant \lambda N(-\frac{1}{2},\frac{1}{2}) + (1-\lambda)N(y,\frac{1}{2})$. Thus, $r_c(1) \leqslant \lambda r_c(1) + (1-\lambda)N(y,\frac{1}{2})$. So we have necessarily $N(y,\frac{1}{2}) = r_c(1)$. By using a symmetrical argument, we have $N(y',\frac{1}{2}) = r_c(1)$ for all $-\frac{1}{2} \leqslant y' \leqslant x$. So by symmetry, we have $C^{d_\infty}_{1/2}(o) \subseteq C^d_{r_c(1)}(o)$ and thus $\mathscr{B}^{d_\infty}_{1/2}(o) = \mathscr{B}^d_{r_c(1)}(o)$. Then by Property 16, there exists $\alpha > 0$ such that $d = \alpha d_\infty$.
($\Leftarrow$) If $d = \alpha d_\infty$, then $C^{d_\infty}_{1/2}(o) = C^d_{r_c(1)}(o)$ and thus $\frac{1}{2}\{-1,1\}^2 \subset (C^d_{r_c(1)}(o) \cap C^{d_\infty}_{1/2}(o))$.

(iii) ($\Rightarrow$) If $r_H(K,1) = r_c(1)$ then there exists $k \in K$ and $p \in \mathbb{Z}^2$ such that $d(k,\mathbb{Z}^2) = d(k,p) = r_c(1)$, then $k \in \mathscr{C}(p,1)$ because d is cellular. So we have necessarily $d_\infty(k,p) = \frac{1}{2}$. Therefore (ii) implies that $k \in (\{(\frac{1}{2},\frac{1}{2})\} \oplus \mathbb{Z}^2)$.
($\Leftarrow$) Let $k \in K \cap (\{(\frac{1}{2},\frac{1}{2})\} \oplus \mathbb{Z}^2)$. Then $d(k,\mathbb{Z}^2) \leqslant r_H(K,1) \leqslant r_c(1)$. But by (i) we have $d(k,\mathbb{Z}^2) = r_c(1)$. $\square$

4.1. Hausdorff discretization and supercover discretization

The supercover discretization operator $\Delta_{\mathscr{SC}}$ is defined by

$$\forall F \in \mathscr{F}'(\mathbb{R}^n), \quad \Delta_{\mathscr{SC}}(F,\rho) = \{p \in \mathscr{D}_\rho \mid F \cap \mathscr{C}(p,\rho) \neq \emptyset\}.$$

In this section, we study the relationship between the supercover discretization and the Hausdorff discretization. We have shown that the supercover discretization is a Hausdorff discretization iff the metric is cellular, and we proved also that the supercover discretization is the maximal Hausdorff discretization iff the metric is proportional to d_∞. The proofs of the results of this section are given in [20] for the special case of compact sets. The proofs for closed sets is similar.

Definition 41. Let $E \subseteq \mathbb{R}^n$, a subset $S \subseteq \mathscr{D}_\rho$ is called a *covering discretization* of E, if $\forall p \in S$, $E \cap \mathscr{C}(p,\rho) \neq \emptyset$ and $E \subseteq \bigcup_{p \in \mathscr{S}} \mathscr{C}(p,\rho)$.

Property 42. *Let* d *be a cellular metric. If* $F \in \mathscr{F}'(\mathbb{R}^n)$ *and* S *is a covering discretization of* F *then* $S \in \mathscr{M}_H(F,\rho)$, *in particular the supercover discretization of* F *is a Hausdorff discretization of* F.

Proof. If $x \in F$, then there exists $s \in S$ such that $x \in \mathscr{C}(s,\rho)$ and thus, $d(x,s) = d(x,\mathscr{D}_\rho) \leqslant r_H(F,\rho)$, so $h_d(F,S) \leqslant r_H(F,\rho)$.

If $s \in S$, then there exists $x \in F$ such that $x \in \mathscr{C}(s,\rho)$ and thus, $d(x,s) = d(x,\mathscr{D}_\rho) \leqslant r_H(F,\rho)$, so $h_d(S,F) \leqslant r_H(F,\rho)$. Therefore $S \in \mathscr{M}_H(F,\rho)$. $\square$

Proposition 7. *Let d be a homogeneous metric, then* $\forall F \in \mathscr{F}'(\mathbb{R}^n)$, $\Delta_H(F,\rho) \subseteq \Delta_{\mathscr{S}\mathscr{C}}(F,\rho) \oplus \Delta_{\mathscr{S}\mathscr{C}}(\mathscr{B}^d_{r_c(\rho)}(o))$.

Theorem 8. *Let d be a metric induced by a norm on $\mathbb{R}^n$.*
- *d is a cellular* $\Leftrightarrow \forall F \in \mathscr{F}'(\mathbb{R}^n), \Delta_{\mathscr{S}\mathscr{C}}(F,\rho) \in \mathscr{M}_H(F,\rho)$.
- $\forall F \in \mathscr{F}'(\mathbb{R}^n), \Delta_H(F,\rho) = \Delta_{\mathscr{S}\mathscr{C}}(F,\rho) \Leftrightarrow \exists \alpha \in \mathbb{R}^+$ *such that* $d = \alpha d_\infty$.

The proofs of Proposition 7 and Theorem 8 are given in [20].

4.2. Hausdorff discretization and Bresenham discretization

In this section, we compare the Hausdorff discretization and the Bresenham discretization [3]. Actually, we prove that the Bresenham discretization of a straight line of $\mathbb{R}^2$ is not always a good discretization relatively to the Hausdorff metric.

This result is an extension of [25] in which we prove the same thing for a segment of $\mathbb{R}^2$.

Definition 43. Let $x, y \in \mathbb{R}^2$, the set $[x,y] = \{tx + (1-t)y \mid 0 \leqslant t \leqslant 1\}$ is called the segment between x and y.

Definition 44 (Bresenham [3]). Let $\mathscr{L} = \{(x, ax+b) \mid x \in \mathbb{R}\}$ be a straight line of $\mathbb{R}^2$ where $a, b \in \mathbb{R}$ such that $0 < |a| \leqslant 1$, $\Delta_{\mathrm{Bres}}(\mathscr{L}) = \{(i, \lfloor ai + b + \frac{1}{2} \rfloor) \mid i \in \mathbb{Z}\}$ where for all $x \in \mathbb{R}$, $\lfloor x \rfloor \in \mathbb{Z}$ and $\lfloor x \rfloor \leqslant x < \lfloor x \rfloor + 1$. $\Delta_{\mathrm{Bres}}(\mathscr{L})$ is called the Bresenham discretization of the straight line $\mathscr{L}$.

Remarks.
- If $p \in \Delta_{\mathrm{Bres}}(\mathscr{L})$ then $\mathscr{L} \cap \mathscr{C}(p,1) \neq \emptyset$.
- $\Delta_{\mathrm{Bres}}(\mathscr{L})$ is functional relatively to the first coordinate: for all i in $\mathbb{Z}$ there exists a unique j in $\mathbb{Z}$ such that (i,j) in $\Delta_{\mathrm{Bres}}(\mathscr{L})$.

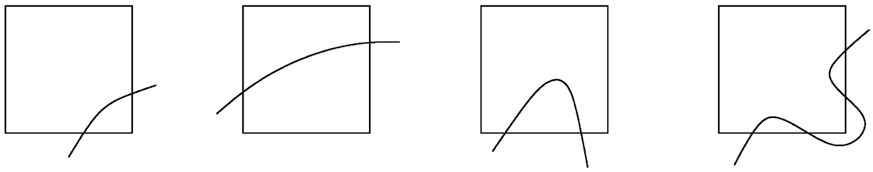
Definition 45. Let d be a metric on $\mathbb{R}^n$ and $F \in \mathscr{F}'(\mathbb{R}^n)$, *the skeleton* of $\Delta_H(F,\rho)$ is the set

$$\mathscr{S}k(F,\rho) = \bigcap_{S \in \mathscr{M}_H(F,\rho)} S.$$

Definition 46. Let F be a subset of $\mathbb{R}^2$ and $\mathscr{S}$ be a square in $\mathbb{R}^2$, we say that F crosses $\mathscr{S}$ if $\exists p, q \in F$ such that p, q belong to two distinct faces of $\mathscr{S}$, the segment $[p,q]$ is not in a face of $\mathscr{S}$ and p, q belong to a same connected component of $F \cap \mathscr{S}$ (i.e. in particular $p \neq q$, $p, q \in (F \cap (\mathscr{S} \setminus int(\mathscr{S}))$ and $[p,q] \cap int(\mathscr{S}) \neq \emptyset$).

p, q are called end points of a crossing of $\mathscr{S}$ by F.

Fig. 5 shows curves that cross a square and curves that do not cross a square.



(a) (b) (c) (d)

Fig. 5. In (a) and (b) the curves cross the square, in (c) and (d) the curves do not.

Property 47. *Let d be a homogeneous metric on $\mathbb{R}^2$, and let $F \in \mathscr{F}'(\mathbb{R}^2)$. If $r_H(F,\rho) < r_c(\rho)$, then*

$$F \text{ crosses } \mathscr{C}(p,\rho) \Rightarrow p \in \mathscr{S}k(F,\rho).$$

Proof. Assume that F crosses $\mathscr{C}(p,\rho)$ and let $S \in \mathscr{M}_H(F,\rho)$. Put $r = r_H(F,\rho) < r_c(\rho)$ and $r' = r/r_c(1)$ ($r' < \rho$). Then Theorem 6 implies that $F \subseteq \bigcup_{q\in S} \mathscr{B}^d_r(q) \subseteq \bigcup_{q\in S} \mathscr{B}^{d_1}_{r'}(q)$. Put $F_p = F \cap \mathscr{C}(p,\rho)$. So if x, y are end points of a crossing of $\mathscr{C}(p,\rho)$ by F, then x, y belong to the same connected component CC_p of F_p and they are on two distinct faces of the square $\mathscr{C}(p,\rho)$.

Assume that $p \notin S$, then by Property 38(i) we have $CC_p \subseteq F_p \subset \bigcup_{q\in\mathscr{V}_4(p)} int(\mathscr{B}^{d_1}_{r''}(q))$ where $int(\mathscr{B}^{d_1}_{r''}(q))$ is the interior of $\mathscr{B}^{d_1}_{r''}(q)$ and $r' < r'' < \rho$. So if $q, q' \in \mathscr{V}_4(p)$ and $q \neq q'$ then $int(\mathscr{B}^{d_1}_{r''}(q)) \cap int(\mathscr{B}^{d_1}_{r''}(q')) = \emptyset$. F crosses $\mathscr{C}(p,\rho)$ and $CC_p \subset \bigcup_{q\in\mathscr{V}_4(p)} int(\mathscr{B}^{d_1}_{r''}(q))$ imply that there exists $q', q'' \in \mathscr{V}_4(p)$ such that $x \in int(\mathscr{B}^{d_1}_{r''}(q'))$, so $CC_p \cap int(\mathscr{B}^{d_1}_{r''}(q')) \neq \emptyset$ and $y \in int(\mathscr{B}^{d_1}_{r''}(q''))$, so $CC_p \cap int(\mathscr{B}^{d_1}_{r''}(q'')) \neq \emptyset$. Thus $q' \neq q''$, because x and y are on two distinct faces of the square $\mathscr{C}(p,\rho)$, which is absurd because for every q, $int(\mathscr{B}^{d_1}_{r''}(q))$ is an open set and CC_p is connected. So we have necessarily $p \in \mathscr{S}k(F,\rho)$. □

Definition 48. Let d be a homogeneous metric on $\mathbb{R}^2$, and let $E = \{(x,y) \in \mathscr{B}^d_{r_c(1)}(o) \mid y = -x+1\}$ where o is the zero vector.

- $\mathscr{A}(d)$ is the point of E of minimal x-coordinate (i.e. if $\mathscr{A}(d) = (x_0, y_0)$, then $x_0 = \min(\{x \mid (x, -x+1) \in E\})$).
- $\mathscr{R}(d)$ is the square with the set of vertices $V(\mathscr{R}) = \{(x_0,x_0), (-x_0,x_0), (-x_0,x_0), (-x_0,-x_0)\}$ (i.e. $\mathscr{R}(d)$ is the convex hull of $V(\mathscr{R})$).

The point $\mathscr{A}(d)$ and the square $\mathscr{R}(d)$ are illustrated in Fig. 6.

Notation. Let d be a metric on $\mathbb{R}^2$, and let $p \in \mathbb{Z}^2$. $\mathscr{N}(p) = \mathbb{R}^2 \setminus (\bigcup_{q\in(\mathbb{Z}^2\setminus\{p\})} \mathscr{B}^d_{r_c(1)}(q))$. $\mathscr{N}(p)$ is illustrated in Fig. 7 for the metric d_2.

Property 49. *Let d be a homogeneous metric on $\mathbb{R}^2$ and let $p \in \mathbb{Z}^2$, then $\mathscr{N}(o)$ is an open subset of $\mathbb{R}^2$ and ($\mathscr{N}(o) = \emptyset \Leftrightarrow \exists\alpha \in \mathbb{R}^+$ such that $d = \alpha d_1$).*

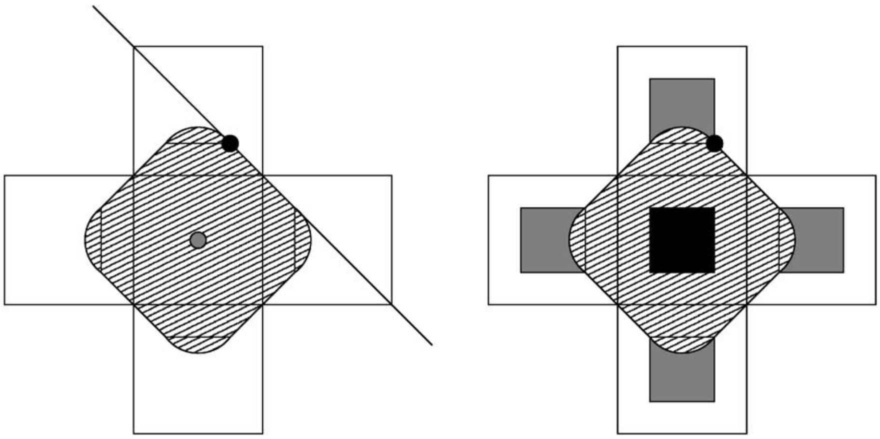


Fig. 6. (a) The black point represents $\mathscr{A}(d)$ for d such that $\mathscr{B}^d_{r_c(1)}(o)$ is the hatched area, where o is the zero vector represented by the grey point. (b) The black square represents $\mathscr{R}(d)$ and the grey squares are the translations of $\mathscr{R}(d)$ to the 4-neighbours of o.

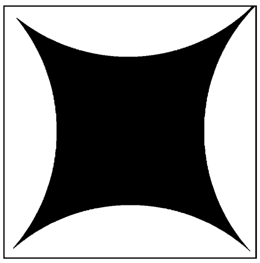


Fig. 7. $\mathscr{N}(o)$ for the metric d_2.

Proof. $\mathscr{N}(o)=\mathbb{R}^2\backslash(\bigcup_{q\in(\mathbb{Z}^2\backslash\{o\})}\mathscr{B}^d_{r_c(1)}(q))$, so $\mathscr{N}(o)\cap\mathscr{C}(q,1)=\emptyset$ for all $q\neq o$, thus $\mathscr{N}(o)\subseteq int(\mathscr{C}(o,1))$.

Property 38 implies that: if $p\neq o$ and $\mathscr{C}(o,1)\cap\mathscr{B}^d_{r_c(1)}(p)\neq\emptyset$ then there exists $q\in\mathscr{V}_4(o)$ such that $\mathscr{C}(o,1)\cap\mathscr{B}^d_{r_c(1)}(p)\subseteq\mathscr{C}(o,1)\cap\mathscr{B}^d_{r_c(1)}(q)$, so $\mathscr{N}(o)=\mathscr{C}(o,1)\backslash(\bigcup_{q\in\mathscr{V}_4(o)}\mathscr{B}^d_{r_c(1)}(q))$ and thus $\mathscr{N}(o)$ is an open set. Therefore $\mathscr{N}(o)=\emptyset$ implies that there exists $q\in\mathscr{V}_4(o)$ such that $o\in\mathscr{B}^d_{r_c(1)}(q)$. But d is a homogeneous metric and $\mathscr{B}^d_{r_c(1)}(q)\subseteq\mathscr{B}^{d_1}_1(q)$ implies that $\mathscr{B}^d_{r_c(1)}(q)=\mathscr{B}^{d_1}_1(q)$. Therefore Property 16 implies that $d=(1/r_c(1))d_1$. $\square$

Remark. $\forall p\in\mathbb{Z}^2$, $\mathscr{N}(p)=\{p\}\oplus\mathscr{N}(o)$ and $\mathscr{N}(p)\subseteq\{p\}\oplus int(\mathscr{R}(d))$, more precisely $\{p\}\oplus int(\mathscr{R}(d))$ is the convex hull of $\mathscr{N}(p)$.

Property 50. *Let d be a homogeneous metric on $\mathbb{R}^2$, and let $\mathscr{L}$ be a straight line of $\mathbb{R}^2$. If $r_H(\mathscr{L},1)=r_c(1)$, and $\mathscr{L}$ crosses $\mathscr{C}(p,1)$, then*

$$p \in \mathscr{S}k(\mathscr{L},1) \;\Leftrightarrow\; \mathscr{L} \cap (\{p\} \oplus int(\mathscr{R}(d))) \neq \emptyset,$$

where $int(\mathscr{R}(d))$ is the interior of the set $\mathscr{R}(d)$.

Proof. ($\Rightarrow$) $r_H(\mathscr{L},1)=r_c(1)$ and $p\in\mathscr{S}k(\mathscr{L},1)$ imply that $\mathscr{L}\cap\mathscr{N}(p)\neq\emptyset$, so $\mathscr{L}\cap(\{p\}\oplus int(\mathscr{R}(d)))\neq\emptyset$ because $\mathscr{N}(p)\subseteq\{p\}\oplus int(\mathscr{R}(d))$.

($\Leftarrow$) $\forall q\in(\mathscr{V}_8(p)\setminus\mathscr{V}_4(p)), \mathscr{B}_1^{d_1}(q)\cap int(\mathscr{C}(p,1))=\emptyset$, then $\mathscr{B}^d_{r_c(1)}(q)\cap(\{p\}\oplus int(\mathscr{R}(d)))=\emptyset$. As a consequence of the minimality of the abscissa of the point $\mathscr{A}(d)$, we have $\forall q,q'\in\mathscr{V}_4(p)$, if $q\neq q'$ then $\mathscr{B}^d_{r_c(1)}(q)\cap\mathscr{B}^d_{r_c(1)}(q')\cap(\{p\}\oplus int(\mathscr{R}(d)))=\emptyset$.

Let $S\in\mathscr{M}_H(\mathscr{L},1)$ and assume that $p\notin S$.

Put $\mathscr{L}_p=\mathscr{L}\cap(\{p\}\oplus int(\mathscr{R}(d)))$. Then we have

- $\mathscr{L}_p\subseteq\bigcup_{q\in\mathscr{V}_4(p)}\mathscr{B}^d_{r_c(1)}(q)$ and
- $\mathscr{L}$ crosses $\{p\}\oplus\mathscr{R}(d)$ because $\mathscr{L}$ crosses $\mathscr{C}(p,1)$ and $\mathscr{L}\cap(\{p\}\oplus int(\mathscr{R}(d)))\neq\emptyset$. So there exists $q_1,q_2\in\mathscr{V}_4(p)$ such that $q_1\neq q_2$ and $\mathscr{L}_p\cap\mathscr{B}^d_{r_c(1)}(q_1)\cap(\{p\}\oplus int(\mathscr{R}(d)))\neq\emptyset$ and $\mathscr{L}_p\cap\mathscr{B}^d_{r_c(1)}(q_2)\cap(\{p\}\oplus int(\mathscr{R}(d)))\neq\emptyset$,

which is absurd because $\mathscr{L}_p$ is connected. So $p\in\mathscr{S}k(\mathscr{L},1)$. □

Theorem 9. *Let d be a homogeneous metric on $\mathbb{R}^2$, and let $\mathscr{L}=\{(x,ax+b)\,|\,x\in\mathbb{R}\}$ be a straight line of $\mathbb{R}^2$ such that $0<|a|\leqslant 1$.*

(i) *If $r_H(\mathscr{L},1)<r_c(1)$, then $\Delta_{\mathrm{Bres}}(\mathscr{L})$ is not a Hausdorff discretization of $\mathscr{L}$ and thus $H_d(\Delta_{\mathrm{Bres}}(\mathscr{L}),\mathscr{L})>r_H(\mathscr{L},1)$.*

(ii) *If $r_H(\mathscr{L},1)=r_c(1)$, then $H_d(\Delta_{\mathrm{Bres}}(\mathscr{L}),\mathscr{L})>r_H(\mathscr{L},1)\Leftrightarrow\exists p,q,r\in\mathbb{Z}^2$ such that $q,r\in\mathscr{V}_4(p)$, $r\in(\mathscr{V}_8(q)\setminus\mathscr{V}_4(q))$ and $\mathscr{L}$ crosses $\{p\}\oplus\mathscr{R}(d)$, $\{q\}\oplus\mathscr{R}(d)$, $\{r\}\oplus\mathscr{R}(d)$.*

Proof. Recall that if $p\in\Delta_{\mathrm{Bres}}(\mathscr{L})$ then $\mathscr{L}\cap\mathscr{C}(p,1)\neq\emptyset$.

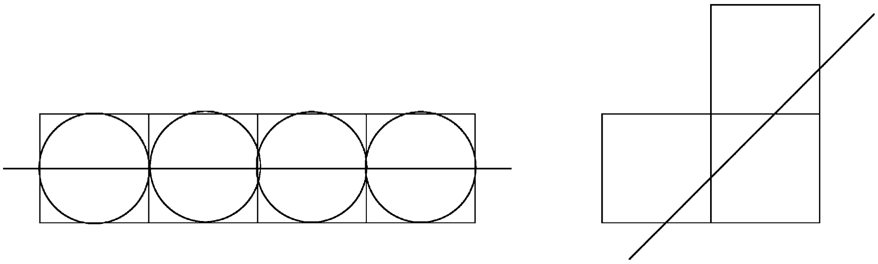
(i) $0<|a|\leqslant 1$ and $r_H(\mathscr{L},1)<r_c(1)$ imply that there exists $q,r\in\Delta_{\mathrm{Bres}}(\mathscr{L})$ such that $q=(x,y)$, $r=(x+1,y+\varepsilon)$ with $\varepsilon\in\{-1,1\}$ $(r\in(\mathscr{V}_8(q)\setminus\mathscr{V}_4(q)))$ and thus $\mathscr{L}$ crosses $\mathscr{C}(q,1)$ and $\mathscr{C}(r,1)$. Therefore $\mathscr{L}$ crosses necessarily $\mathscr{C}(p,1)$ where $p=(x+\varepsilon',y+(1-\varepsilon')\varepsilon)$ for $\varepsilon'\in\{0,1\}$. So $\mathscr{L}$ crosses $\mathscr{C}(p,1)$, $\mathscr{C}(q,1)$, $\mathscr{C}(r,1)$ with $q,r\in\mathscr{V}_4(p)$ and $r\in(\mathscr{V}_8(q)\setminus\mathscr{V}_4(q))$.
So by Property 47 we have $p,q,r\in\mathscr{S}k(\mathscr{L},1)$, therefore if S is a Hausdorff discretization of $\mathscr{L}$ then S is not functional relatively to the first coordinate. Thus $\Delta_{\mathrm{Bres}}(\mathscr{L})$ is not a Hausdorff discretization of $\mathscr{L}$ because $\Delta_{\mathrm{Bres}}(\mathscr{L})$ is functional relatively to the first coordinate.

(ii) ($\Leftarrow$) If $\exists p,q,r\in\mathbb{Z}^2$ such that $q,r\in\mathscr{V}_4(p)$, $r\in(\mathscr{V}_8(q)\setminus\mathscr{V}_4(q))$ and $\mathscr{L}$ crosses $\{p\}\oplus\mathscr{R}(d)$, $\{q\}\oplus\mathscr{R}(d)$, $\{r\}\oplus\mathscr{R}(d)$. Property 50 implies that $p,q,r\in\mathscr{S}k(\mathscr{L},1)$, then if S is a Hausdorff discretization of $\mathscr{L}$, then S is not functional relatively to the first coordinate. Thus $H_d(\Delta_{\mathrm{Bres}}(\mathscr{L}),\mathscr{L})>r_H(\mathscr{L},1)$ because $\Delta_{\mathrm{Bres}}(\mathscr{L})$ is functional relatively to the first coordinate.

($\Rightarrow$) $H_d(\Delta_{\mathrm{Bres}}(\mathscr{L}),\mathscr{L})>r_H(\mathscr{L},1)$, $0<|a|\leqslant 1$ and $r_H(\mathscr{L},1)=r_{\mathrm{c}}(1)$ implies that there exists $q,r\in\Delta_{\mathrm{Bres}}(\mathscr{L})$ such that $q=(x,y)$, $r=(x+1,y+\varepsilon)$ with $\varepsilon\in\{-1,1\}$ ($q\in(\mathscr{V}_8(r)\backslash\mathscr{V}_4(r))$) and $\mathscr{L}$ crosses $\{p\}\oplus\mathscr{R}(d)$ where $p=(x+\varepsilon',y+(1-\varepsilon')\varepsilon)$ for $\varepsilon'\in\{0,1\}$.
In order to simplify the proof, we assume that $a>0$ and $q=(-1,0)$, then $r=(0,1)$ and ($p=(0,0)$ or $p=(-1,1)$).
We will prove in *two steps* that $\mathscr{L}$ crosses $\{(0,1)\}\oplus\mathscr{R}(d)$ and $\{(-1,0)\}\oplus\mathscr{R}(d)$. We assume that $p=(0,0)$, the case $p=(-1,1)$ is proved using exactly the same techniques.
There exists, h such that $\mathscr{R}(d)=\{x\in\mathbb{R}^2\mid d_\infty(o,x)\leqslant h\}$. (*The side of* $\mathscr{R}(d)$ *has a length of* $2h$, *so* $0\leqslant h\leqslant\frac{1}{2}$.)

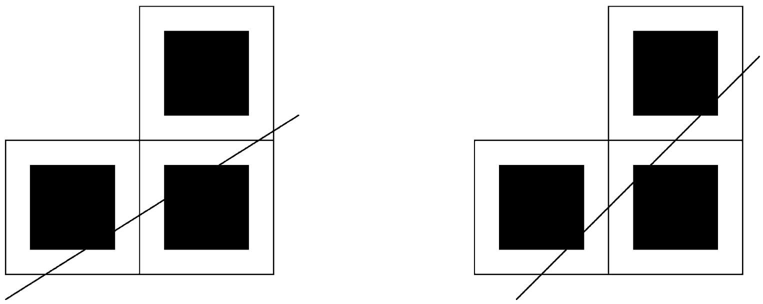
Step 1: We will prove in this step that $\mathscr{L}$ intersects necessarily two adjacent sides of $\mathscr{R}(d)$.
Assume that $\mathscr{L}$ intersects two opposite sides of $\mathscr{R}(d)$:
(1) Assume that the sides of $\mathscr{R}(d)$ intersected by $\mathscr{L}$ are the south and the north sides. Then we have $-ah+b\leqslant -h<ah+b$ and $-ah+b<h\leqslant ah+b$ which imply that $b<ah\leqslant\frac{1}{2}$ because $0<a\leqslant 1$ and $0\leqslant h\leqslant\frac{1}{2}$, which is absurd because $(1,0)\in\Delta_{\mathrm{Bres}}(\mathscr{L})$ implies that $\frac{1}{2}\leqslant b<\frac{3}{2}$.
(2) Assume that the sides of $\mathscr{R}(d)$ intersected by $\mathscr{L}$ are the east and the west sides. Then we have $-h\leqslant -ah+b<h$, $-h<ah+b\leqslant h$ which imply that $b<h\leqslant\frac{1}{2}$, which is absurd because $(1,0)\in\Delta_{\mathrm{Bres}}(\mathscr{L})$ imply that $\frac{1}{2}\leqslant b<\frac{3}{2}$. So $\mathscr{L}$ intersects two adjacent sides of $\mathscr{R}(d)$.

Step 2: $\mathscr{L}$ intersects two adjacent sides of $\mathscr{R}(d)$ and $(0,1)\in\Delta_{\mathrm{Bres}}(\mathscr{L})$ implies that $\mathscr{L}$ intersects the east and north sides of $\mathscr{R}(d)$.
($*$) We will prove that $\mathscr{L}$ crosses $\{(0,1)\}\oplus\mathscr{R}(d)$.
So $\mathscr{L}$ intersects the east and north sides of $\mathscr{R}(d)$ and $(0,1)\in\Delta_{\mathrm{Bres}}(\mathscr{L})$ imply that $\frac{1}{2}\leqslant b<\frac{3}{2}$, $-h\leqslant -ah+b<h$, $-ah+b<h\leqslant ah+b$ and $0\leqslant h\leqslant\frac{1}{2}$:
(1) $-ah+b<h$ and $0\leqslant h\leqslant\frac{1}{2}$ imply that $-ah+b<1-h$,
(2) $-ah+b<h$ and $-\frac{1}{2}<1-b\leqslant\frac{1}{2}$ imply that $-ah-b+1<h$ because $1-b\leqslant b$, so $1-h<ah+b$.
Thus (1) and (2) imply that $\mathscr{L}$ intersects the south side of $\{(0,1)\}\oplus\mathscr{R}(d)$. But, $\mathscr{L}$ crosses $\mathscr{C}((0,1),1)$ then $\mathscr{L}$ crosses $\{(0,1)\}\oplus\mathscr{R}(d)$.
($**$) We will prove that $\mathscr{L}$ crosses $\{(-1,0)\}\oplus\mathscr{R}(d)$.
So $\mathscr{L}$ intersects the east and north sides of $\mathscr{R}(d)$ and $(-1,0),(0,1)\in\Delta_{\mathrm{Bres}}(\mathscr{L})$ imply that $\frac{1}{2}\leqslant b<\frac{3}{2}$, $-h\leqslant -ah+b<h$, $-ah+b<h\leqslant ah+b$, $0\leqslant h\leqslant\frac{1}{2}$ and $|a-b|\leqslant\frac{1}{2}$:
(1) $-ha+b<h$ implies that $-ha-|b-a|<h-a$, so $-ha-b<h-a$ because $\frac{1}{2}\leqslant b<\frac{3}{2}$ and $0<a\leqslant 1$.
(2) $-h<ah-b$ and $0\leqslant h\leqslant\frac{1}{2}$ imply that $-h<a(1-h)-b$.
Thus (1) and (2) is equivalent to $-h<(-1+h)a+b<h$, which is equivalent to $\mathscr{L}$ intersects the west side of $\{(-1,0)\}\oplus\mathscr{R}(d)$. But $\mathscr{L}$ crosses $\mathscr{C}((-1,0),1)$, then $\mathscr{L}$ crosses $\{(-1,0)\}\oplus\mathscr{R}(d)$.



(a) (b)

Fig. 8. (a) and (b) illustrate a different case of Theorem 9 where $r_H(\mathscr{L},1) < r_c(1)$ for the metric d_2.



(a) (b)

Fig. 9. (a) and (b) illustrate a different case of Theorem 9 where $r_H(\mathscr{L},1) = r_c(1)$.

So we have $q, r \in \mathscr{V}_4(p)$, $r \in (\mathscr{V}_8(q) \setminus \mathscr{V}_4(q))$ and $\mathscr{L}$ crosses $\{p\} \oplus \mathscr{R}(d)$, $\{q\} \oplus \mathscr{R}(d)$ and $\{r\} \oplus \mathscr{R}(d)$. $\square$

The different cases of Theorem 9 are illustrated in Figs. 8 and 9. The relationship between Hausdorff discretizations and Bresenham discretization of straight line $\mathscr{L}$ depend on positions of $\mathscr{L}$ relatively to the squares $\mathscr{C}(p,1)$ and $\{p\} \oplus \mathscr{R}(d)$ for $p \in \mathbb{Z}^2$. Statement (i) of Theorem 9 implies that the Bresnham discretization is 'almost surely' not a Hausdorff discretization.

5. Topological properties of Hausdorff discretization in the real plane

In this section, we study some topological properties of Hausdorff discretizations for homogeneous metrics in the real plane $\mathbb{R}^2$. Actually, we prove that a Hausdorff discretization set of a connected closed set is 8-connected and its maximal Hausdorff discretization is 4-connected.

The topological properties of the supercover discretization have been studied by several people. Schmitt [21] shows that, given a compact set K of $\mathbb{R}^2$, under some conditions on K, points can be removed from its supercover discretization $\Delta_{\mathscr{SC}}(K)$, in such a way that, for the remaining subset S of points, $\bigcup_{p\in S}\mathscr{C}(p,1)$ is homotopically equivalent to K. Latecki et al. [16] give sufficient conditions on a compact set K under which $\Delta_{\mathscr{SC}}(K)$ is topologically equivalent to K.

Property 51. *Let d be a homogeneous metric, and let $F\in\mathscr{F}'(\mathbb{R}^n)$. If F is connected then $\forall S\in\mathscr{M}_H(F,\rho), S$ is 8-connected, and $\Delta_H(F,\rho)$ is 4-connected.*

Proof.

- Let $S\in\mathscr{M}_H(F,\rho)$ and assume that S is not 8-connected. Let S_1 be a 8-connected component of S and let $S_2=S\backslash S_1$, then $S_2\neq\emptyset$. Put $F_1=\bigcup_{p\in S_1}\mathscr{B}^d_{r_H(F,\rho)}(p)$ and $F_2=\bigcup_{p\in S_2}\mathscr{B}^d_{r_H(F,\rho)}(p)$. Thus F_1,F_2 are closed sets and $F\subseteq F_1\cup F_2$. So F connected implies that $F\cap F_1\cap F_2\neq\emptyset$. Thus there exists $p_1\in S_1$ and $p_2\in S_2$ such that $\mathscr{B}^d_{r_H(F,\rho)}(p_1)\cap\mathscr{B}^d_{r_H(F,\rho)}(p_2)\neq\emptyset$. Then Property 38 implies that $p_2\in\mathscr{V}_8(p_1)$ because $r_H(F,\rho)\leqslant r_c(\rho)$, which is absurd because S_1 is a 8-connected component of S. Therefore S is 8-connected.
- Assume that $\Delta_H(F,\rho)$ is not 4-connected and let S_1 be a 4-connected component of S and let $S_2=S\backslash S_1$, then $S_2\neq\emptyset$. Let $F_1=\bigcup_{p\in S_1}\mathscr{B}^d_{r_H(F,\rho)}(p)$ and $F_2=\bigcup_{p\in S_2}\mathscr{B}^d_{r_H(F,\rho)}(p)$. So F_1,F_2 are closed sets and $F\subseteq F_1\cup F_2$. Thus F connected implies that $F\cap F_1\cap F_2\neq\emptyset$. Then there exists $p_1\in S_1$, $p_2\in S_2$ such that $\mathscr{B}^d_{r_H(F,\rho)}(p_1)\cap\mathscr{B}^d_{r_H(F,\rho)}(p_2)\neq\emptyset$.
  Therefore two cases are possible:
  - $r_H(F,\rho)<r_c(\rho)$ and thus $p_2\in\mathscr{V}_4(p_1)$, which is absurd because S_1 is a 4-connected component of S.
  - $r_H(F,\rho)=r_c(\rho)$. Thus $p_2\in(\mathscr{V}_8(p_1)\backslash\mathscr{V}_4(p_1))$. So if $x\in F\cap\mathscr{B}^d_{r_c(\rho)}(p_1)\cap\mathscr{B}^d_{r_c(\rho)}(p_2)$ then $x\in\mathscr{B}^{d_1}_\rho(p_1)\cap\mathscr{B}^{d_1}_\rho(p_2)$, and thus there exists $p_3\in(\mathscr{V}_4(p_1)\cap\mathscr{V}_4(p_2))$ such that $x\in\mathscr{C}(p_3,\rho)$, as $\Delta_{\mathscr{SC}}(F,\rho)\subseteq\Delta_H(F,\rho)$ we have $p_3\in\Delta_H(F,\rho)$, which is absurd because S_1 is a 4-connected component of S.

  Therefore $\Delta_H(F,\rho)$ is 4-connected. $\square$

The converse of the last property is not true for every closed set.

Consider for example the set $E=\{(x,1/x)\,|\,x\in]0,1]\}\cup\{(-x,1/x)\,|\,x\in]0,1]\}$ which is a closed set of $\mathbb{R}^2$ relatively to any distance induced by a norm. It is easy to see that, $\forall\rho>0$, $\forall S\in\mathscr{M}_H(E,\rho)$, S is 8-connected, but E is not connected.

We prove in the following property, that the last property has a converse if F is a compact set.

Property 52. *Let d be a homogeneous metric, and let $K\in\mathscr{K}(\mathbb{R}^2)$, and assume that there exists $\rho_0>0$ such that for every $\rho<\rho_0$ there exists $S^\rho\in\mathscr{M}_H(K,\rho)$ such that S^ρ is 8-connected, then K is connected.*

Proof. Assume that, there exists $\rho_0>0$ such that for every $\rho<\rho_0$ there exists $S^\rho\in\mathscr{M}_H(K,\rho)$ such that S^ρ is 8-connected, but K is not connected. So there are two non-empty closed sets K_1, K_2 such that $K = K_1 \cup K_2$ and $K_1 \cap K_2 = \emptyset$. That K is a compact set implies that K_1, K_2 are compact sets.

Let $r = \min(\{d(x_1,x_2) \mid x_1 \in K_1,\ x_2 \in K_2\})$. So $r > 0$ because K_1, K_2 are a disjoint compact sets. Let $\rho < r/4r_c(1)$ and $S^\rho \in \mathscr{M}_H(K,\rho)$ such that S^ρ is 8-connected. So $K \subseteq \bigcup_{p \in S^\rho} \mathscr{B}^d_{r_H(K,\rho)}(p)$ and there exists $p_1, p_2 \in S^\rho$ such that $\mathscr{B}^d_{r_H(K,\rho)}(p_1) \cap K_1 \neq \emptyset$ and $\mathscr{B}^d_{r_H(K,\rho)}(p_2) \cap K_2 \neq \emptyset$ and $p_2 \in \mathscr{V}_8(p_1) \cup \{p_1\}$. Let $x_1 \in \mathscr{B}^d_{r_H(K,\rho)}(p_1) \cap K_1$ and $x_2 \in \mathscr{B}^d_{r_H(K,\rho)}(p_2) \cap K_2$. Then $d(x_1,x_2) \leqslant 4r_c(\rho) = 4\rho r_c(1) < r$, which is absurd.

Therefore, if there exists $\rho_0 > 0$, such that for every $\rho < \rho_0$, there exists $S^\rho \in \mathscr{M}_H(K,\rho)$ such that S^ρ is 8-connected, then K is connected. $\square$

6. Conclusion

We have introduced a new framework for the discretization of non-empty closed sets, based on the Hausdorff distance. We have proved the convergence (*in the Hausdorff metric sense*) of the discretization to the original object when the resolution of the discrete space converges to zero. Based on this new discretization, we have proposed a new discretization of operators on the set of non-empty closed sets. We refined the study of the Hausdorff discretization for homogeneous metrics and we investigated the relationship between Hausdorff discretizations and covering discretizations. We compared the Bresenham discretization and Hausdorff discretizations of a straight line. Actually, we have proved that the Bresenham discretization is not always minimizing the Hausdorff metric. It is easy to prove that the standard discrete straight line [18] is always a Hausdorff discretization. The conditions on the straight line for which the Bresenham discretization is not a Hausdorff discretization can be given in a more explicit way by using the Fourier–Motzkin algorithm [7, 15].

Finally, we studied some topological properties of Hausdorff discretizations. In [24], we have refined the study of the transfer of topological properties for Hausdorff discretizations in $\mathbb{R}^2$. We have proved that a Hausdorff discretization 'preserves' the homotopy for a large class of closed sets called r-convex sets, for a subclass of homogeneous metrics; and under some general conditions on the metric, every Hausdorff discretization of K is 'homeomorphic' to K for every compact set K in subclass of the class of r-convex sets.

Further investigations will be needed on:
- the differential properties of Hausdorff discretization;
- its extension to grey-level images;
- the other alternatives for discretization of geometrical and morphological operators.

References

[1] E. Andrès, Standard cover: a new class of discrete primitives, Internal report, IRCOM, Université de Poitiers, 1998.
[2] M.F. Barnsley, Fractals Everywhere, 2nd Edition, Academic Press, New York, 1993.

[3] J.E. Bresenham, Algorithms for computer control of digital plotter, IBM Systems J. 4 (1) (1965) 25–30.
[4] H. Bussemann, The Geometry of Geodesics, Academic Press, New York, 1955.
[5] D. Cohen-Or, A. Kaufman, Fundamentals of surface voxelization, Graphical Models Image Process. 57 (6) (1995) 453–461.
[6] S. Duval, M. Tajine, Digital geometry and fractal geometry, 5th Coll. DGCI'95, 1995, pp. 93–106.
[7] J.B.J. Fourier, Solution d'une question particulière du calcul des inégalités, Oeuvre II, Paris, 1826, pp. 317–328.
[8] K. Goebel, W.A. Kirk, Topics in metric fixed point theory, Cambridge University Press, Cambridge, 1990.
[9] F. Hausdorff, Set Theory, Chelsea, New York, 1962.
[10] H.J.A.M. Heijmans, Morphological discretization, in: U. Eckhardt, et al. (Eds.), Geometrical Problems in Image Processing, Akademie Verlag, Berlin, 1991, pp. 99–106.
[11] H.J.A.M. Heijmans, Discretization of morphological operators, J. Visual Commun. Image Representation 3 (2) (1992) 182–193.
[12] H.J.A.M. Heijmans, Morphological Image Operators, Academic Press, New York, 1994.
[13] H.J.A.M. Heijmans, A. Toet, Morphological sampling, Comput. Vision, Graphics Image Process: Image Understanding 54 (3) (1991) 384–400.
[14] J.G. Hocking, G.S. Young, Topology, Dover Publications Inc., New York, 1988.
[15] H.W. Kuhn, Solvability and consistency for linear equations and inequalities, Amer. Math. Monthly 63 (1956) 217–232.
[16] L.J. Latecki, C. Conrad, A. Gross, Preserving topology by a digitalization process, J. Math. Imaging Vision 8 (1998) 131–159.
[17] G. Matheron, Random Sets and Integral Geometry, Wiley, New York, 1975.
[18] J.-P. Reveillès, Géométrie discrète, calcul en nombres entiers et algorithmique, Thèse de Doctorat d'Etat, Université Louis Pasteur, Strasbourg, France, December 1991.
[19] C. Ronse, M. Tajine, Discretization in Hausdorff space, J. Math. Imaging Vision 12 (3) (2000) 219–242.
[20] C. Ronse, M. Tajine, Hausdorff discretization for cellular distances, and its relation to cover and supercover discretization (to be revised for JVCIR), 2000.
[21] M. Schmitt, Digitalization and Connectivity, ISMM'98, Kluwer Academic Publishers, Dordrecht, 1998, pp. 91–98.
[22] J. Serra, Image Analysis and Mathematical Morphology, Academic Press, London, 1982.
[23] J. Stoer, C. Witzgall, Convexity and Optimization in Finite Dimension I, Springer, Berlin, 1970, pp. 131–133.
[24] M. Tajine, C. Ronse, Topological properties of Hausdorff discretizations, ISMM'2000, Palo Alto (USA), Kluwer Academic Publishers, 2000, pp. 41–50.
[25] M. Tajine, D. Wagner, C. Ronse, Hausdorff Discretization and its Comparison with other Discretization Schemes, DGCI'99, Paris, Lecture Notes in Computer Sciences, Vol. 1568, Springer, Berlin, 1999, pp. 399–410.
[26] D. Wagner, Distance de Hausdorff et problème discret-continu, Mémoire de D.E.A, M.Sc. Dissertation, Université Louis Pasteur, Strasbourg (France), June 1997.
[27] D. Wagner, M. Tajine, C. Ronse, An Approach to Discretization Based on the Hausdorff Metric. I, ISMM'98, Kluwer Academic Publishers, Dordrecht, 1998, pp. 67–74.